APPENDIX: MATHEMATICAL INDUCTION AND OTHER FORMS OF PROOF

When you are done with your homework you should be able to…

- Use the Principle of Mathematical Induction to prove statements involving a positive integer *n*
- Prove by contradiction that a mathematical statement is true
- Use a counterexample to show that a mathematical statement is false

Mathematical Induction

Mathematical Induction is a method of mathematical ___proof___ used to establish a given statement for all ___natural___ numbers. It is a form of ___direct___ proof. It is done in ___2___ steps. The first step, known as the ___base___ case, is to prove the given statement for the first natural number. The second step, known as the ___inductive___ step, is to prove that the given statement for any one natural number ___implies___ the given statement for the next natural number.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer *n*. If

1. P_1 is true, and
2. for every ___positive___ integer *k*, the truth of ___P_K___ implies the truth of ___P_{K+1}___ then the statement P_n must be true for all positive ___integers___ *n*.

Example 1: Use mathematical induction to prove the formula for every positive integer *n*.

$$S_n = 1^3+2^3+3^3+\cdots+n^3=\frac{n^2\left(n+1\right)^2}{4}$$

① Base case

$$S_k = 1^3+2^3+3^3+\cdots+k^3 = \underbrace{\frac{k^2(k+1)^2}{4}}_{}$$

$$S_1 = 1^3 = 1$$

$$1 \overset{?}{=} \frac{1^2(1+1)^2}{4}$$

$$1 \overset{?}{=} \frac{4}{4}$$

$$1 = 1 \checkmark$$

So S_1 is true.

② Inductive step

$$S_{k+1} = \underbrace{1^3+2^3+3^3+\cdots+k^3}_{S_k} + (k+1)^3$$

$$S_{k+1} = S_k + (k+1)^3$$

$$S_{k+1} = \underbrace{\frac{k^2(k+1)^2}{4}}_{} + (k+1)^3\cdot\frac{4}{4}$$

$$\frac{(k+1)^2\left[(k+1)+1\right]^2}{4} = \frac{(k+1)^2\left[k^2+\overbrace{4(k+1)}^{4k+4}\right]}{4}$$

$$\frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4} \checkmark$$

$\therefore$ for $n \in \mathbb{Z}^+$, $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$, QED

PROOF BY CONTRADICTION

In mathematical logic, proof by contradiction is described by the following equivalence:

_p__ implies _q__ if and only if _not_____ _q__ implies _not_____ _p_.

One way to prove that _q_ is a true statement is to assume that q is _not__ true.

If this leads you to a statement that you know is _false__________, then you have proved that _q__ must be _true_______.

Example 2: Use proof by contradiction to prove the statement.

a. $\overbrace{\text{If } a \text{ and } b \text{ are real numbers and } 1<a<b,}^{p}$ then $\overbrace{\frac{1}{a}>\frac{1}{b}}^{q}$.

Proof: Suppose $\frac{1}{a} \le \frac{1}{b}$. Then $ab\left(\frac{1}{a}\right) \le ab\left(\frac{1}{b}\right)$, and $b \le a$. This contradicts $1<a<b$. $\therefore$ by proof by contradiction, $\frac{1}{a} > \frac{1}{b}$. QED

b. If a is a real number and $0<a<1$, then $a^2<a$.

Proof: Suppose $a^2 \ge a$. Then $\frac{a^2}{a} \ge \frac{a}{a}$. Which gives us $a \ge 1$, which contradicts $0<a<1$. $\therefore$ by proof by contradiction, $a^2 < a$. QED

USING COUNTEREXAMPLES

Example 3: Use a counterexample to show that the statement is false.

a. The product of two irrational numbers is irrational.

$\sqrt{2}$ is irrational and $(\sqrt{2})(\sqrt{2}) = 2$ which is rational.

So the statement is false.

b. If f is a polynomial function and $f(a) = f(b)$, then $a = b$.

$f(x) = x^2$ is a polynomial function. $f(2) = 4$ and

$f(-2) = 4$, but $2 \neq -2$. So the statement is false.

Section 1.1: INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

When you are done with your homework you should be able to…

- Recognize a linear equation in *n* variables
- Find a parametric representation of a solution set
- Determine whether a system of linear equations is consistent of inconsistent
- Use back-substitution and Gaussian elimination to solve a system of linear equations

WARM-UP: Solve the system.

a.

$$\begin{aligned} -x+8y &= 3 \quad R_1 \\ 4y &= 2 \quad R_2 \end{aligned}$$

Isolate y in R_2 and sub. into R_1

$$y=\frac{1}{2} \rightarrow -x+8\left(\frac{1}{2}\right)=3$$

$$-x = -1$$

$$x = 1$$

$\{(1, \frac{1}{2})\}$

system has independent equations, an it is a consistent system.

b.

$$\begin{aligned} 3x+y-z &= -4 \\ -2y+4z &= 0 \\ z &= -1 \end{aligned}$$

row-echelon form

1)

$$-2y+4(-1)=0$$

$$y=-2$$

2)

$$3x+(-2)-(-1)=-4$$

$$x=-1$$

$\{(-1,-2,-1)\}$, consistent system with independent equations.

DEFINITION OF A LINEAR EQUATION IN *n* VARIABLES

A linear equation in *n* variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

The coefficients $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the constant term b is a real number. The number a_1 is the leading coefficient, and x_1 is the leading variable.

*Linear equations have no products or roots of variables and no variables involved in transcendental functions.

Example 1: Give an example of a linear equation in three variables.

$$x_1 + 5x_2 - \frac{1}{3}x_3 = 17$$

SOLUTIONS AND SOLUTION SETS

A solution of a linear equation in *n* variables is a sequence of *n* real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3, \ldots, x_n = s_n$$

into the equation. The set of all solutions of a linear equation is called its solution set, and when you have found this set, you have satisfied the equation. To describe the entire solution set of a linear equation, use a parametric representation.

Example 2: Solve the linear equation $x_1 + x_2 = 10$.

$x_1 = 10 - x_2$

Let $x_2 = t$

$x_1 = 10 - t$

$$\boxed{x_1 = 10 - t,\ x_2 = t,\ t \in \mathbb{R}}$$

Example 3: Solve the linear equation $2x_1 - x_2 + 5x_3 = -1$.

$2x_1 = x_2 - 5x_3 - 1$

$x_1 = \frac{1}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}$

Let $x_2 = s$, $x_3 = t$

$$\boxed{\begin{array}{l} x_1 = \frac{1}{2}s - \frac{5}{2}t - \frac{1}{2},\ x_2 = s,\ x_3 = t, \\ s,t \in \mathbb{R}. \end{array}}$$

SYSTEMS OF LINEAR EQUATIONS IN *n* VARIABLES

A system of linear equations in *n* variables is a set of *m* equations, each of which is linear in the same *n* variables.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n &= b_m
\end{aligned}$$

SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

A solution of a system of linear equations is a __sequence_______________ of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the __system___________________.

Example 4: Graph the following linear systems and determine the solution(s), if a solution exists.

a.

$x-y=12$

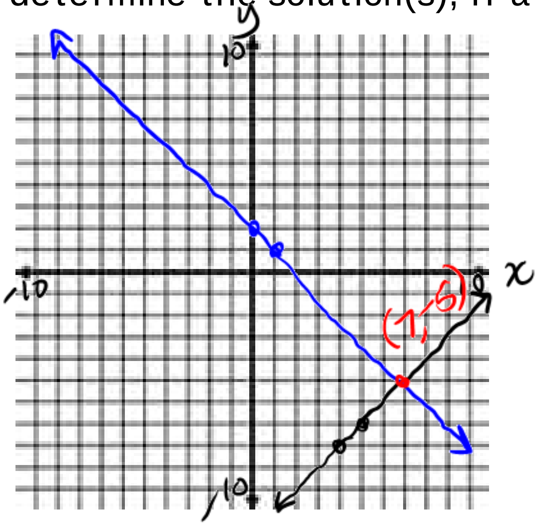
$x+y=2$

$(5,-7)$, $(4,-8)$

$(1,1)$, $(0,2)$

$\{(7,-5)\}$

consistent and independent



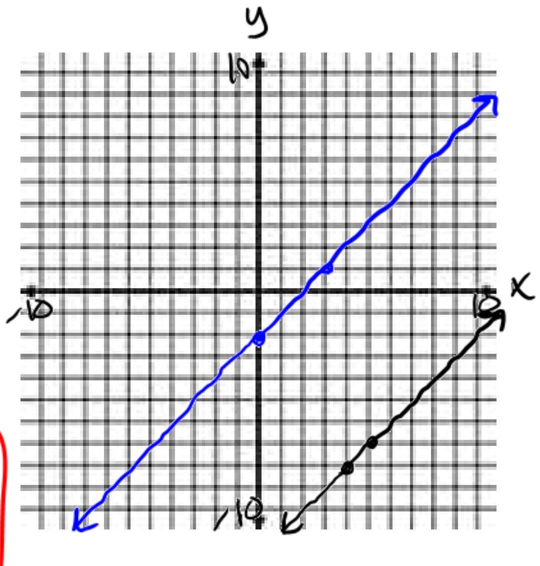
b.

$x-y=12$ → $y=x-12$

$x-y=2$ → $y=x-2$

$(3,1)$, $(0,-2)$

$\emptyset$ or $\{\ \}$, inconsistent and independent



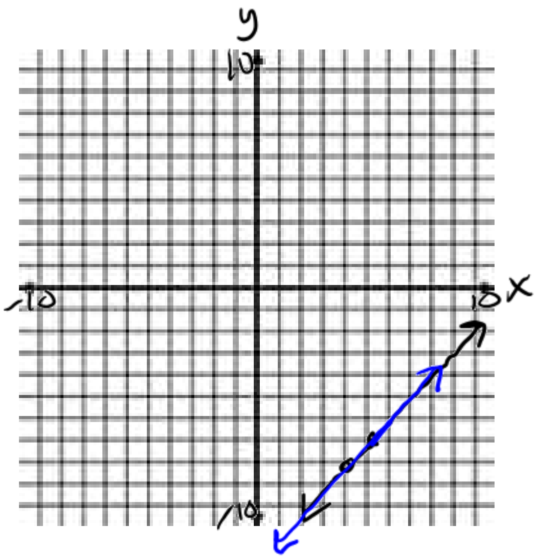
c.

$x-\ \ y=12$

$2x-2y=24$

$\{(x,y): x,y\in\mathbb{R},\ x-y=12\}$

consistent and dependent.



NUMBER OF SOLUTIONS OF A SYSTEM OF EQUATIONS

For a system of linear equations, precisely one of the following is true.

1. The system has exactly one solution. (consistent system).
2. The system has infinitely many solutions (consistent system)
3. The system has no solution (inconsistent system).

OPERATIONS THAT PRODUCE EQUIVALENT SYSTEMS

Each of the following operations on a system of linear equations produces an equivalent system.

1. Add two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

The idea is to get the system into row-echelon form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

Example 5: Solve the system of linear equations.

a.

$$\begin{aligned} \frac{2}{3}x_1 + \frac{1}{6}x_2 &= 0 \quad R_1 \\ 4x_1 + x_2 &= 0 \quad R_2 \end{aligned}$$

$\downarrow -6R_1 + R_2 \to R_2$

$$\begin{aligned} \frac{2}{3}x_1 + \frac{1}{6}x_2 &= 0 \\ 0 + 0 &= 0 \end{aligned}$$

infinitely many solutions

$$4x_1 + x_2 = 0$$

$$x_1 = -\frac{x_2}{4}$$

Let $x_2 = -4t$

$$\boxed{x_1 = t, \ x_2 = -4t, \ t \in \mathbb{R}}$$

b.

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \quad R_1 \\ -x_1 + 3x_2 - 2x_3 &= 8 \quad R_2 \\ 4x_1 + x_2 &= 4 \quad R_3 \end{aligned}$$

$\downarrow$ $R_2 \leftrightarrow R_3$

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \quad R_1 \\ 4x_1 + x_2 &= 4 \quad R_2 \\ -x_1 + 3x_2 - 2x_3 &= 8 \quad R_3 \end{aligned}$$

$\downarrow 2R_1 + R_3 \to R_3$

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \quad R_1 \\ 4x_1 + x_2 &= 4 \quad R_2 \\ x_1 + x_2 &= 12 \quad R_3 \end{aligned}$$

$\to -R_2 + R_3 \to R_3$

$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 4x_1 + x_2 &= 4 \\ -3x_1 &= 8 \end{aligned}\right\}$$

Row-echelon form

Gaussian form (sort of)

$$x_1 = -\frac{8}{3}, \ x_2 = \frac{44}{3}, \ x_3 = \frac{58}{3}$$

$\left\{\left(-\frac{8}{3}, \frac{44}{3}, \frac{58}{3}\right)\right\}$

consistent

c.

$$\begin{aligned} 5x_1 - \ \ 3x_2 + 2x_3 &= 3 \\ 2x_1 + \ \ 4x_2 - \ \ x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

Section 1.2: GAUSSIAN ELIMINATION AND GAUSS-JORDAN ELIMINATION

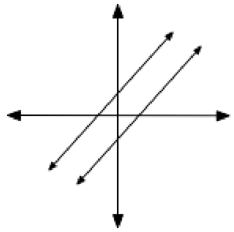
When you are done with your homework you should be able to…

- Determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations
- Solve a homogeneous system of linear equations

TYPES OF SOLUTIONS

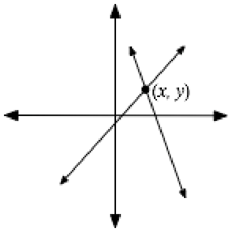
2 Equations, 2 Variables

Parallel Lines



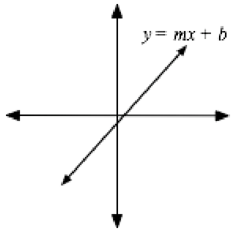
No points in common.
Solution: $\varnothing$

Intersecting Lines



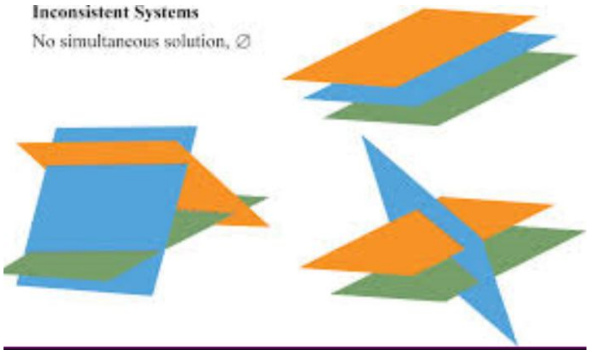
One point in common.
Solution: (x, y)

Coincident Lines



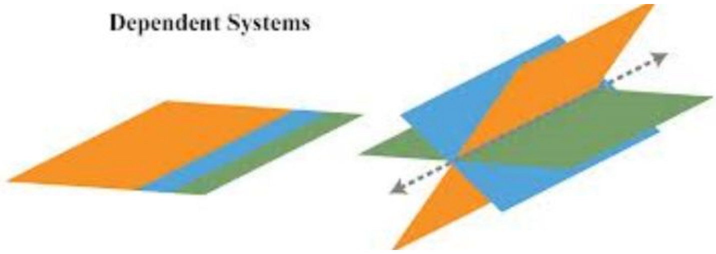
Infinitely many points in common.
Solution: $\{(x, y): y = mx + b\}$

3 Equations, 3 Variables



Inconsistent Systems

No simultaneous solution, $\varnothing$



Dependent Systems

DEFINITION OF A MATRIX

If m and n are positive integers, an $m \times n$ matrix (read m by n) matrix is a rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which each entry, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

*The entry a_{ij} is located in the ith row and the jth column. The index i is called the row subscript because it identifies the row in which the entry lies, and the index j is called the column subscript because it identifies the column in which the entry lies.

**A matrix with m rows and n columns is said to be of size $m \times n$.

When $m = n$, the matrix is called square of order n and the entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal entries.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A is 3×3, A is square of order 3

$a_{11} = 1$, $a_{23} = 6$

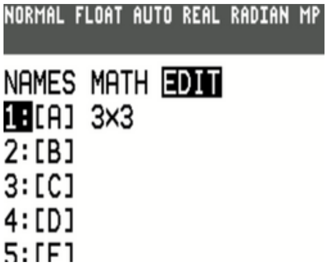
Example 1: Consider the system of equations we solved in 1.1.

$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

a. What is the coefficient matrix? What is the size of the coefficient matrix?

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 2 & 4 & -1 \\ 1 & -11 & 4 \end{bmatrix}$$

A is 3x3



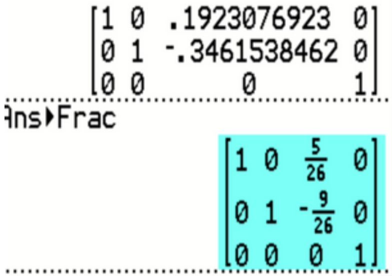
```
NORMAL FLOAT AUTO REAL RADIAN MP
NAMES MATH EDIT
1:[A] 3x3
2:[B]
3:[C]
4:[D]
5:[E]
```

2nd x^{-1}

b. What is the augmented matrix?

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 \quad + \tfrac{5}{26}x_3 &= 0 \\ x_2 \quad - \tfrac{9}{26}x_3 &= 0 \\ 0 &= 1 \end{aligned}$$



```
[1 0  .1923076923 0]
[0 1 -.3461538462 0]
[0 0            0 1]
Ans▸Frac
[1 0  5/26 0]
[0 1 -9/26 0]
[0 0     0 1]
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
NAMES MATH EDIT
8↑Matr▸list(
9:List▸matr(
0:cumSum(
A:ref(
B:rref(
C:rowSwap(
D:row+(
E:*row(
F:*row+(
```

$\boxed{\{\ \},\ \text{inconsistent system}}$

ELEMENTARY ROW OPERATIONS

1. Add two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.
4. Interchange (swap) any 2 rows.

ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM

A matrix in row-echelon form has the following properties.

1. Any rows consisting entirely of Zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).
3. For two successive nonzero rows, the leading 1 in the higher row is farther to the right than the leading 1 in the lower row.

A matrix in row-echelon form is in reduced row-echelon form when every column that has a leading 1 has Zeros in every position above and below its leading 1.

GAUSSIAN ELIMINATION WITH BACK SUBSTITUTION

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back substitution to find the solution.

Example 2: Solve the system using Gaussian elimination with back substitution.

$$5x_1 - 3x_2 + 2x_3 = 3$$
$$2x_1 + 4x_2 - x_3 = 7$$
$$x_1 - 11x_2 + 4x_3 = 3$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$$\Big\downarrow -2R_3 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 1 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$$\Big\downarrow -R_1 + 5R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 1 \\ 0 & -52 & 18 & 12 \end{array}\right]$$

$$\Big\downarrow 2R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 1 \\ 0 & 0 & 0 & 14 \end{array}\right] \longrightarrow \begin{array}{rl} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 26x_2 - 9x_3 &= 1 \\ 0 &= 14 \end{array}$$

$$\boxed{\{\ \},\ \text{inconsistent system}}$$

Example 3: Solve the system using Gauss-Jordan elimination.

$$\begin{aligned} x_1 \quad + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - \ x_3 &= 0 \end{aligned}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 1 & 1 & -5 & 3 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\downarrow$$

$-1R_1 + R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\downarrow$$

$-2R_1 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & -1 & 3 & -2 \end{array}\right]$$

$$\downarrow$$

$R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 \quad -2x_3 = 1 \rightarrow x_1 = 1+2t$$

$$x_2 \ -3x_3 = 2 \rightarrow x_2 = 2+3t$$

$$x_3 = t$$

$$\boxed{x_1 = 1+2t,\ x_2 = 2+3t,\ x_3 = t,\ t \in \mathbb{R}}$$

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of equations in which each of the __constant__ terms is zero are called __homogeneous__. A homogeneous system of *m* equations in *n* variables has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n &= 0 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n &= 0
\end{aligned}
$$

****Homogenous linear systems either have the __trivial__ solution, or __infinitely__ __many__ solutions**

Example 4: Solve the homogeneous linear system corresponding to the given coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

augmented matrix: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$

Let $x_3 = s$, $x_4 = t$

$$
\begin{aligned}
x_1 \qquad &= 0 \\
x_2 + x_3 &= 0 \rightarrow x_2 = -s
\end{aligned}
$$

$$\boxed{\begin{array}{l} x_1 = 0, x_2 = -s, \\ x_3 = s, x_4 = t, \\ s, t \in \mathbb{R} \end{array}}$$

THEOREM 1.1: THE NUMBER OF SOLUTIONS OF A HOMOGENEOUS SYSTEM

Every homogeneous system of linear equations is __consistent__. If the system has fewer equations than variables, then it must have __infinitely__ __many__ solutions.

Section 1.3: APPLICATIONS OF SYSTEMS OF LINEAR EQUATIONS

When you are done with your homework you should be able to…

- Set up and solve a system of equations to fit a polynomial function to a set of data points
- Set up and solve a system of equations to represent a network

POLYNOMIAL CURVE FITTING

Suppose n points in the xy-plane

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

represent a collection of data and you are asked to find a

polynomial function of degree n-1

whose graph passes through the specified points. This procedure is called

polynomial curve fitting.

If all x-coordinates are distinct, then there is precisely one polynomial function of degree $n-1$ (or less) that fits the n points. To solve for the n coefficients of $p(x)$, Substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$
\begin{aligned}
a_0 + a_1x_1 + a_2x_1^{\,2} + \cdots a_{n-1}x_1^{\,n-1} &= y_1 \\
a_0 + a_1x_2 + a_2x_2^{\,2} + \cdots a_{n-1}x_2^{\,n-1} &= y_2 \\
a_0 + a_1x_3 + a_2x_3^{\,2} + \cdots a_{n-1}x_3^{\,n-1} &= y_3 \\
&\vdots \\
a_0 + a_1x_n + a_2x_n^{\,2} + \cdots a_{n-1}x_n^{\,n-1} &= y_n
\end{aligned}
$$

Example 1: Determine the polynomial function whose graph passes through the points, and graph the polynomial function, showing the given points.

$(2,4),(3,4),(4,4)$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(2) = a_0 + a_1(2) + a_2(2)^2 = 4 \rightarrow 1a_0 + 2a_1 + 4a_2 = 4$

$p(3) = a_0 + a_1(3) + a_2(3)^2 = 4 \rightarrow 1a_0 + 3a_1 + 9a_2 = 4$

$p(4) = a_0 + a_1(4) + a_2(4)^2 = 4 \rightarrow 1a_0 + 4a_1 + 16a_2 = 4$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 4 \\ 1 & 4 & 16 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \begin{array}{l} a_0 = 4 \\ a_1 = 0 \\ a_2 = 0 \end{array}$$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(x) = 4 + 0x + 0x^2$

$\boxed{p(x) = 4}$

Example 2: The table shows the U.S. population figures for the years 1940, 1950, 1960, and 1970. (Source: U.S. Census Bureau)

Year	1940 → 0	1950 → 10	1960 → 20	1970 → 30
Population (in millions)	132	151	179	203

a. Find a cubic polynomial that fits these data and use it to estimate the population in 1980.

Let x represent the # of years after 1940.

Let y represent the population in millions.

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$p(0) = 1a_0 = 132$$

$$p(10) = 1a_0 + 10a_1 + 100a_2 + 1000a_3 = 151$$

$$p(20) = 1a_0 + 20a_1 + 400a_2 + 8000a_3 = 179$$

$$p(30) = 1a_0 + 30a_1 + 900a_2 + 27000a_3 = 203$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 132 \\ 1 & 10 & 100 & 1000 & 151 \\ 1 & 20 & 400 & 8000 & 179 \\ 1 & 30 & 900 & 27000 & 203 \end{array}\right] \rightarrow \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 132 \\ 0 & 1 & 0 & 0 & 1.016 \\ 0 & 0 & 1 & 0 & 0.110 \\ 0 & 0 & 0 & 1 & -0.002 \end{array}\right]$$

$a_0 = 132$

$a_1 \doteq 1.016$

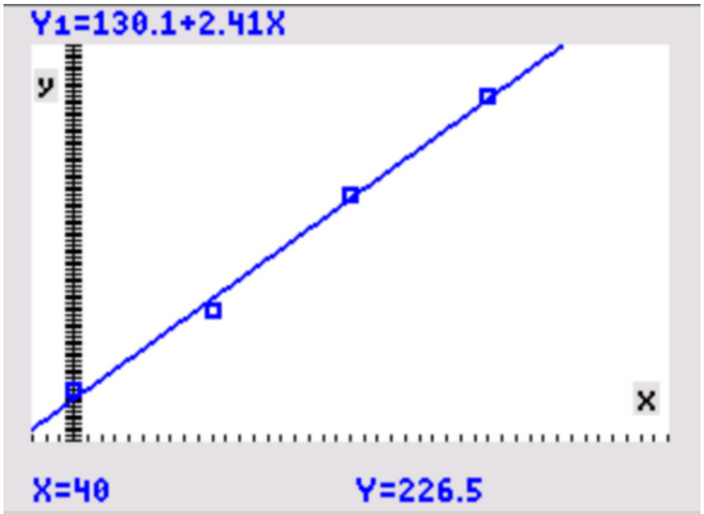
$a_2 \doteq 0.110$

$a_3 \doteq -0.002$

$$\boxed{p(x) = 132 + 1.016x + 0.110x^2 - 0.002x^3}$$

b. The actual population in 1980 was 227 million. How does your estimate ~~to~~ compare?

$$p(40) = 132 + 1.016(40) + 0.110(40)^2 - 0.002(40)^3 \doteq 221$$



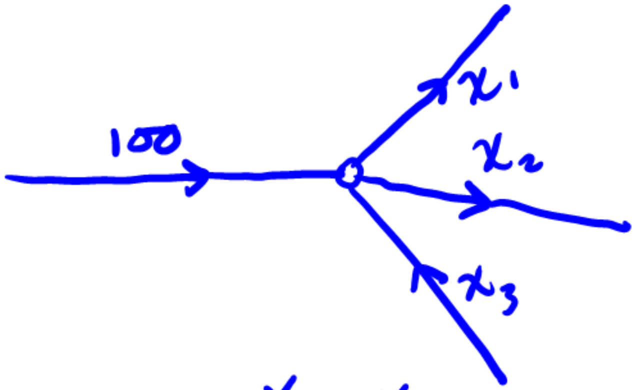
The data is linear!

So $\hat{y}(x) = 130.1 + 2.41x$ gives us a better prediction model.

$\hat{y}(40) = 226{,}500{,}000$ which is closer to the actual population in 1980 of 227,000,000.

NETWORK ANALYSIS

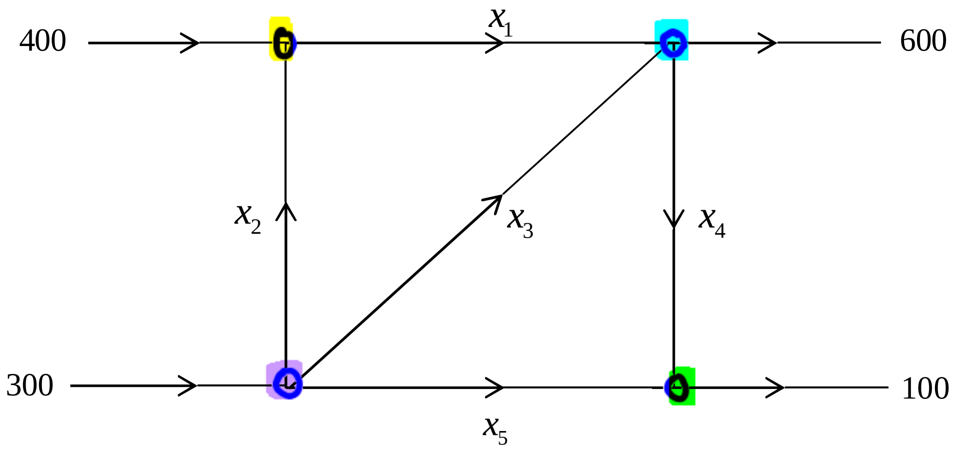
Networks composed of ____branches____ and ____junctions____ are used as models in fields like economics, traffic analysis, and electrical engineering. In a network model you assume that the total ____flow____ into a ____junction____ is equal to the total flow ____out____ of the junction.



$$100 + x_3 = x_1 + x_2$$

$$x_1 + x_2 - x_3 = 100$$

Example 3: The figure shows the flow of traffic through a network of streets.



a. Solve this system for x_i, $i = 1, 2, \ldots, 5$.

$$\left.\begin{aligned} 400 + x_2 &= x_1 \\ x_1 + x_3 &= 600 + x_4 \\ x_4 + x_5 &= 100 \\ 300 &= x_2 + x_3 + x_5 \end{aligned}\right\} \rightarrow \begin{aligned} x_1 - x_2 &= 400 \\ x_1 + x_3 - x_4 &= 600 \\ x_4 + x_5 &= 100 \\ x_2 + x_3 + x_5 &= 300 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_3 + x_5 &= 700 \rightarrow x_1 = 700 - x_3 - x_5 \\ x_2 + x_3 + x_5 &= 300 \rightarrow x_2 = 300 - x_3 - x_5 \\ x_4 + x_5 &= 100 \rightarrow x_4 = 100 - x_5 \end{aligned}$$

$x_1 = 700 - s - t$, $x_2 = 300 - s - t$, $x_3 = s$, $x_4 = 100 - t$, $x_5 = t$

$s, t \in \mathbb{R}$

b. Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

$x_1 = 700 - 0 - 100 = 600$

$x_2 = 300 - 0 - 100 = 200$

$x_3 = 0$

$x_4 = 100 - 100 = 0$

$x_5 = 100$

c. Find the traffic flow when $x_3 = x_5 = 100$.

$x_1 = 700 - 100 - 100 = 500$

$x_2 = 300 - 100 - 100 = 100$

$x_3 = 100$

$x_4 = 100 - 100 = 0$

$x_5 = 100$

Section 2.1: OPERATIONS WITH MATRICES

When you are done with your homework you should be able to…

π Determine whether two matrices are equal
π Add and subtract matrices and multiply a matrix by a scalar
π Multiply two matrices
π Use matrices to solve a system of equations
π Partition a matrix and write a linear combination of column vectors

Matrices are represented in the following ways:

① Uppercase letter → A, B, or C

② representative element → $A \rightarrow [a_{ij}]$

③ Rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

DEFINITION OF EQUALITY OF MATRICES

Two matrices $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are __equal________ when they have the same __size__________ __m×n_________ and __$a_{ij} = b_{ij}$________ for __$1 \le i \le m$________ and __$1 \le j \le n$__________.

Example 1: Are matrices A and B equal? Please explain.

$$A=\begin{bmatrix}1 & -1 & 3 & 8\end{bmatrix}$$

1x4

$$B=\begin{bmatrix}1\\-1\\3\\8\end{bmatrix}$$

4x1

No A and B are different sizes.

Example 2: Find *x* and *y*.

$$\begin{bmatrix} 2x-1 & 4 \\ 3 & y^3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 3 & \frac{1}{8} \end{bmatrix}$$

$2x-1 = -5$

$\boxed{x = -2}$

$y^3 = \frac{1}{8}$

$\boxed{y = \frac{1}{2}}$

A matrix that has only one column is called a column matrix or column vector.

A matrix that has only one row is called a row matrix or row vector.

Boldface lowercase letters often designate column matrices and row matrices.

We will use $\vec{a} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $A = [\vec{a}_1 \,|\, \vec{a}_2]$

DEFINITION OF MATRIX ADDITION

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their sum is the $m \times n$ matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

The sum of two matrices of different sizes is undefined.

DEFINITION OF SCALAR MULTIPLICATION

If $A=\left[a_{ij}\right]$ is an $m\times n$ matrix and c is a scalar, then the scalar multiple of A by c is the mxn matrix given by
$cA=\left[ca_{ij}\right]$

You can use -A to represent the scalar product (-1)A. If A and B are of the same size, then $A-B$ represents the sum of A and -B.

Example 3: For the matrices $\underset{3\times3}{A}=\begin{bmatrix}1 & -3 & 6\\ 2 & 0 & 2\\ -2 & 8 & -1\end{bmatrix}$ and $\underset{3\times3}{B}=\begin{bmatrix}5 & 2 & 7\\ -1 & 9 & -4\\ -3 & 0 & 1\end{bmatrix}$, find

a. $A+B=\begin{bmatrix}1+5 & -3+2 & 6+7\\ 2+-1 & 0+9 & 2+-4\\ -2+-3 & 8+0 & -1+1\end{bmatrix}=\begin{bmatrix}5+1 & 2+-3 & 7+6\\ -1+2 & 9+0 & -4+2\\ -3+-2 & 0+8 & 1+-1\end{bmatrix}=B+A$

$$=\boxed{\begin{bmatrix}6 & -1 & 13\\ 1 & 9 & -2\\ -5 & 8 & 0\end{bmatrix}}$$

hmm... maybe matrix addition is commutative.

b. $2A-B=\begin{bmatrix}-3 & -8 & 5\\ 5 & -9 & 8\\ -1 & 16 & -3\end{bmatrix}$

DEFINITION OF MATRIX MULTIPLICATION

If $A=\left[a_{ij}\right]$ is an $m\times n$ matrix and $B=\left[b_{ij}\right]$ is an $n\times p$ matrix, then the product AB is an $m\times p$ matrix.

$$AB = C = \left[c_{ij}\right]$$

where

$$c_{ij} = \sum_{K=1}^{n} a_{iK}b_{Kj} = a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

To find an entry in the *i*th row and the *j*th column of the product AB, multiply the entries in the ith row of A by the corresponding entries in the jth column of B and and then add the results.

Example 4: Find the product AB, where

$A=\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $B=\begin{bmatrix} -12 & 7 \end{bmatrix}$

2×1 $\quad$ 1×2

$$AB = C$$

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix}\begin{bmatrix} -12 & 7 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \boxed{\begin{bmatrix} -48 & 28 \\ 36 & -21 \end{bmatrix}}$$

$c_{11} = (4)(-12) = -48$

$c_{12} = (4)(7) = 28$

$c_{21} = (-3)(-12) = 36$

$c_{22} = (-3)(7) = -21$

Example 5: Consider the matrices A and B.

$A=\begin{bmatrix}-1 & 3\\ 11 & 13\end{bmatrix}$ and $B=\begin{bmatrix}-4 & 4\\ 6 & 13\end{bmatrix}$

2×2 2×2

a. Find $A+B$.

$$A+B=\begin{bmatrix}-5 & 7\\ 17 & 26\end{bmatrix}$$

c. Find $B+A$.

$$B+A=\begin{bmatrix}-5 & 7\\ 17 & 26\end{bmatrix}$$

b. Find AB.

$$AB=\begin{bmatrix}(-1)(-4)+(3)(6) & (-1)(4)+(3)(13)\\ (11)(-4)+(13)(6) & (11)(4)+(13)(13)\end{bmatrix}$$

$$\boxed{AB=\begin{bmatrix}22 & 35\\ 34 & 213\end{bmatrix}}$$

d. Find BA.

$$\begin{bmatrix}-4 & 4\\ 6 & 13\end{bmatrix}\begin{bmatrix}-1 & 3\\ 11 & 13\end{bmatrix}=$$

$$\begin{bmatrix}(-4)(-1)+(4)(11) & (-4)(3)+(4)(13)\\ (6)(-1)+(13)(11) & (6)(13)+(13)(13)\end{bmatrix}=$$

$$\boxed{\begin{bmatrix}48 & 40\\ 137 & 187\end{bmatrix}}=BA$$

e. Is matrix addition commutative?

Yep!

f. Is matrix multiplication commutative?

Hell NO!

Example 6: Multiply.

$$\underset{3x3}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}}\underset{3 \times 1}{\overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \overset{A\vec{x}}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}}$$

size after mult.

dim. need to match to mult.

SYSTEMS OF LINEAR EQUATIONS

The system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or equivalently,

$$A\vec{x} = \vec{b}$$

Example 7: Write the system of equations in the form $A\mathbf{x}=\mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

$$\begin{aligned}2x_1+3x_2 &= 5\\ x_1+4x_2 &= 10\end{aligned}$$

$$\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}5\\ 10\end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

$$\left[\begin{array}{cc|c}2 & 3 & 5\\ 1 & 4 & 10\end{array}\right] \xrightarrow{A\leftrightarrow B} \left[\begin{array}{cc|c}1 & 4 & 10\\ 2 & 3 & 5\end{array}\right] \xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{cc|c}1 & 4 & 10\\ 0 & -5 & -15\end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2\to R_2} \left[\begin{array}{cc|c}1 & 4 & 10\\ 0 & 1 & 3\end{array}\right] \xrightarrow{-4R_2+R_1\to R_1} \left[\begin{array}{cc|c}1 & 0 & -2\\ 0 & 1 & 3\end{array}\right]$$

$$x_1 = -2$$
$$x_2 = 3$$

$\boxed{\{(-2,3)\}, \text{ consistent and independent.}}$

PARTITIONED MATRICES

$$A=\begin{array}{c}\begin{matrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{matrix}\\ \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}\end{array} \qquad \mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$$

LINEAR COMBINATIONS

The matrix product $A\mathbf{x}$ is a linear combination of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ that form the coefficient matrix A.

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{m3} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

Example 8: Write the column matrix $\mathbf{b}$ as a linear combination of the columns of A.

$$A = \begin{array}{c} \begin{matrix} \vec{a}_1 & a_2 \end{matrix} \\ \begin{bmatrix} -3 & 5 \\ 3 & 4 \\ 4 & 8 \end{bmatrix} \end{array} \qquad \mathbf{b} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

$$x_1\begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 5 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

inconsistent system

$$\left[\begin{array}{cc|c} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & 8 & 32 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\vec{b}$ cannot be expressed as a linear combination of the columns of A, so the system is inconsistent.

Example 9: Find the products AB and BA for the diagonal matrices.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$AB = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hmmm...

maybe the product of diagonal matrices is commutative.

Example 10: Let A and B be matrices such that the product of AB is defined. Show that if A has two identical rows, then the corresponding two rows of AB are also identical.

Proof: Let $A=[a_{ij}]$, $B=[b_{ij}]$ $\ni$ $a_{ij}, b_{ij} \in \mathbb{R}$, A, B are 3x3.

$a_{11} = a_{21}$, $a_{12} = a_{22}$, $a_{13} = a_{23}$. Let $C = AB$.

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

QED

Section 2.2: PROPERTIES OF MATRIX OPERATIONS

When you are done with your homework you should be able to…

- Use the properties of matrix addition, scalar multiplication, and zero matrices
- Use the properties of matrix multiplication and the identity matrix
- Find the transpose of a matrix

THEOREM 2.1: PROPERTIES OF MATRIX ADDITION AND SCALAR MULTIPLICATION

If A, B, and C are $m\times n$ matrices, and c and d are scalars, then the following properties are true.

1. $A+B=$ ___$B+A$___ Commutative property of addition

Pf: Let $A=[a_{ij}]$, $B=[b_{ij}]$, A,B are $m \times n$, $a_{ij}, b_{ij} \in \mathbb{R}$.

$A+B=[a_{ij}+b_{ij}]=[b_{ij}+a_{ij}]$ (real #'s are commut.) and

$[b_{ij}+a_{ij}]=B+A$. //

2. $A+(B+C)=$ ___$(A+B)+C$___ Associative property of addition

3. $(cd)A=$ ___$c(dA)$___ Associative property of multiplication

Pf: Let $A=[a_{ij}]$, $a_{ij}, c, d \in \mathbb{R}$. $(cd)A = cd[a_{ij}] = [(cd)a_{ij}]$.

Now $[(cd)a_{ij}] = [c(da_{ij})]$ (real #'s are assoc.). So

$[c(da_{ij})] = c[da_{ij}] = c(dA)$. //

4. $1A=$ ___A___ Multiplicative Identity (Scalar)

5. $c(A+B)=$ ___$cA+cB$___ Distributive property

Pf: $A=[a_{ij}]$, $B=[b_{ij}]$, $a_{ij}, b_{ij}, c \in \mathbb{R}$. $c(A+B) = c([a_{ij}]+[b_{ij}])$

$= c[a_{ij}+b_{ij}] = [c(a_{ij}+b_{ij})] = [ca_{ij}+cb_{ij}]$ ($\mathbb{R}$ are dist.)

$= [ca_{ij}]+[cb_{ij}] = c[a_{ij}]+c[b_{ij}] = cA+cB$. //

6. $(c+d)A=$ ___$cA+dA$___ Distributive property

Example 1: For the matrices below, $c=-2$, and $d=5$

$$A=\begin{bmatrix}-3 & 5\\ 3 & 4\\ 4 & 8\end{bmatrix} \qquad B=\begin{bmatrix}1 & 1\\ 2 & 7\\ 6 & 9\end{bmatrix} \qquad C=\begin{bmatrix}-7 & 1\\ -2 & 3\\ 11 & 2\end{bmatrix}$$

a. $c(A+C)$ $= -2\begin{bmatrix}-10 & 6\\ 1 & 7\\ 15 & 10\end{bmatrix} = \boxed{\begin{bmatrix}20 & -12\\ -2 & -14\\ -30 & -20\end{bmatrix}}$

b. cdB $= -10\begin{bmatrix}1 & 1\\ 2 & 7\\ 6 & 9\end{bmatrix} = \boxed{\begin{bmatrix}-10 & -10\\ -20 & -70\\ -60 & -90\end{bmatrix}}$

c. $cA-(B+C)$

THEOREM 2.2: PROPERTIES OF ZERO MATRICES

If A is an $m \times n$ matrix, and c is a scalar, then the following properties are true.

1. $A + O_{mn} =$ __A______ additive identity
2. $A + (-A) =$ __O_{mn}____ additive inverse
3. If $cA = O$, then __$c = 0$ or $A = O_{mn}$____________________.

Example 2: Solve for X in the equation, given

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

a. $X = 3A - 2B$

$$X = \begin{bmatrix} -6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}$$

b. $2A + 4B = -2X$

$$-A - 2B = X$$

$$X = \begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix}$$

THEOREM 2.3: PROPERTIES OF MATRIX MULTIPLICATION

If A, B, and C are matrices (with sizes such that the given matrix products are defined), and c is a scalar, then the following properties are true.

1. $A(BC) =$ (AB)C ______________________ Associative property of multiplication

Let $A = [a_{ij}]$, and is $m \times n$. $B = [b_{ij}]$, B is $n \times p$. $C = [c_{ij}]$, C is $p \times q$.

$a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$. $D = ABC$.

$$D = \sum_p \sum_q A_{ip} B_{pq} C_{qj} = \sum_q \left(\sum_p A_{ip} B_{pq} \right) C_{qj} = \sum_q (AB)_{iq} C_{qj} = (AB)C$$

$$D = \sum_p \sum_q A_{ip} B_{pq} C_{qj} = \sum_p A_{ip} \left(\sum_q B_{pq} C_{qj} \right) = \sum_p A_{ip} (BC)_{pq} = A(BC)$$

//

2. $A(B+C) =$ AB + AC __________________ Distributive property of multiplication

3. $(A+B)C =$ AC + BC __________________ Distributive property of multiplication

4. $c(AB) = (cA)B =$ A(cB) _______________

Example 3: Show that $AC = BC$, even though $A \neq B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 12 & -6 & 3 \\ 16 & -8 & 4 \\ 4 & -2 & 1 \end{bmatrix} \qquad BC = \begin{bmatrix} 12 & -6 & 3 \\ 16 & -8 & 4 \\ 4 & -2 & 1 \end{bmatrix}$$

Example 4: Show that $AB = O_{mn}$, even though $A \neq O$ and $B \neq O$.

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

THEOREM 2.4: PROPERTIES OF THE IDENTITY MATRIX

If A is an $m \times n$ matrix, then the following properties are true.

1. $AI_n =$ _A_____ 2. $I_m A =$ _A_____

square matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$AI_2 = \begin{bmatrix} 1+0 & 0+2 \\ 3+0 & 0+4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$I_2 A =$ LTS

e o t
f u
t d
e
n
t

A is $m \times n$
I_n is $n \times n$
$AI_n = A$
I_m is $m \times m$
$I_m A = A$

THEOREM 2.5: NUMBER OF SOLUTIONS OF A LINEAR SYSTEM

For a system of linear equations, precisely one of the following is true.

1. The system has exactly _one______ solution.
2. The system has _infinitely__________ many solutions.
3. The system has _no______ solution.

Proof: In text

in octave : A'

THE TRANSPOSE OF A MATRIX

The transpose of a matrix is denoted A^T and is formed by writing its columns as rows.

Example 5: Find the transpose of the matrix.

$$A = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$$

Symmetric matrices

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$(B^T)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$$

THEOREM 2.5: PROPERTIES OF TRANSPOSES

If A and B are matrices (with sizes such that the given matrix operations are defined), and c is a scalar, then the following properties are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A+B)^T = A^T + B^T$ Transpose of a sum

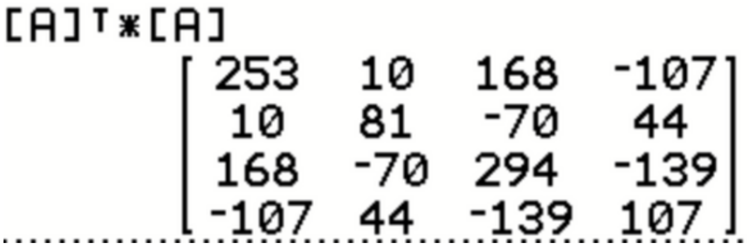
Let A, B be mxn. $A = [a_{ij}]$, $B = [b_{ij}]$, $a_{ij}, b_{ij} \in \mathbb{R}$.

$$(A+B)^T = ([a_{ij}] + [b_{ij}])^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = A^T + B^T \checkmark$$

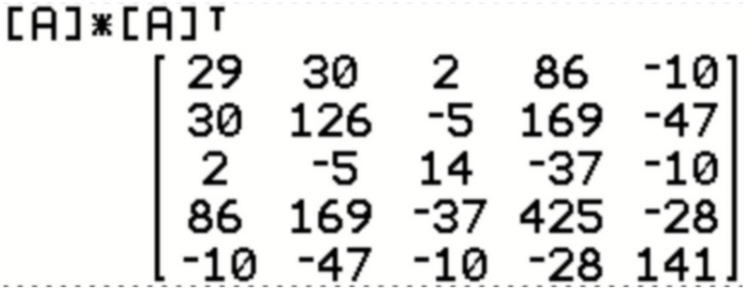
3. $(cA)^T = cA^T$ Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ Transpose of a product

Example 6: Find a) $A^T A$ and b) AA^T. Show that each of these products is symmetric.

$$A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$$



```
[A]T*[A]
      [ 253   10   168  -107]
      [ 10    81   -70   44 ]
      [ 168  -70   294  -139]
      [-107   44  -139   107]
```



```
[A]*[A]T
      [29   30    2    86  -10]
      [30   126  -5   169  -47]
      [2    -5    14  -37  -10]
      [86   169  -37  425  -28]
      [-10  -47  -10  -28  141]
```

Example 7: A square matrix is called skew-symmetric when $A^T = -A$. Prove that if A and B are skew-symmetric matrices, then $A + B$ is skew-symmetric.

Pf: $A^T = -A, \; B^T = -B.$

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ &= -A + (-B) \\ &= -(A+B) \; // \end{aligned}$$

Section 2.3: THE INVERSE OF A MATRIX

When you are done with your homework you should be able to…

- π Find the inverse of a matrix (if it exists)
- π Use properties of inverse matrices
- π Use an inverse matrix to solve a system of linear equations

$$2x = 5$$
$$\left(\tfrac{1}{2}\right)2x = \left(\tfrac{1}{2}\right)5$$
$$1x = \tfrac{5}{2}$$
$$x = 5/2$$

Let $2 = m$, $5 = n$

$$m^{-1}mx = m^{-1}n$$
$$1x = m^{-1}n$$
$$x = m^{-1}n$$

DEFINITION OF THE INVERSE OF A MATRIX

An $n \times n$ matrix A is invertible________ or nonsingular____________ when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity________ matrix of order n. The matrix B is called the (multiplicative__________) inverse__________ of A. A matrix that does not have an inverse is called noninvertible or singular__________.

***Nonsquare matrices do not have inverses__________.**

Example 1: For the matrices below, show that B is the inverse of A.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $AB = BA = I_2$, B is the inverse of A.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$
$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

THEOREM 2.7: UNIQUENESS OF AN INVERSE

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted A^{-1}.

Proof: Since A is invertible, $\overset{\text{there exists}}{\exists}$ a $B \ni AB = I_n$. (defn)

Suppose A has another inverse, C, $\ni AC = I_n = CA$.

$AB = I$

$C(AB) = CI$

$(CA)B = C$

$IB = C$

$\rightarrow B = C$. ∴ The inverse of a matrix is unique. //

FINDING THE INVERSE OF A MATRIX BY GAUSS-JORDAN ELIMINATION

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \; I]$. This process is called adjoining matrix I to matrix A.
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A \; I]$. The result will be the matrix $[I \; A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

Example 2: Find the inverse of the matrix (if it exists), by solving the matrix equation $AX = I$.

$$A = \begin{bmatrix} 12 & 3 \\ 5 & -2 \end{bmatrix}$$

$$[A\ I] = \left[\begin{array}{cc|cc} 12 & 3 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

$\downarrow -5R_1 + 12R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 12 & 3 & 1 & 0 \\ 0 & -39 & -5 & 12 \end{array}\right]$$

$\downarrow 13R_1 + R_2 \to R_1$

$$\left[\begin{array}{cc|cc} 156 & 0 & 8 & 12 \\ 0 & -39 & -5 & 12 \end{array}\right]$$

$\frac{1}{156}R_1 \to R_1,\ -\frac{1}{39}R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/39 & 1/13 \\ 0 & 1 & 5/39 & -4/13 \end{array}\right]$$

$$= [I\ A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2/39 & 1/13 \\ 6/39 & -4/13 \end{bmatrix}}$$

Example 3: Find the inverse of the matrix (if it exists).

a.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$$

THEOREM 2.8: PROPERTIES OF INVERSE MATRICES

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $\left(A^{-1}\right)^{-1} =$ A

2. $\left(A^k\right)^{-1} =$ $\underbrace{A^{-1} \cdot A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}} = \left(A^{-1}\right)^k$

3. $(cA)^{-1} =$ $\rightarrow \frac{1}{c}A^{-1}$

4. $\left(A^T\right)^{-1} =$ $\left(A^{-1}\right)^T$

THEOREM 2.9: THE INVERSE OF A PRODUCT

If A and B are invertible matrices of order n, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Example 4: Use the inverse matrices below for the following problems.

$$A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

a. $(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} -10/77 + 6/77 & 5/77 + 4/77 \\ -6/77 - 3/77 & 3/77 - 2/77 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -4/77 & 9/77 \\ -9/77 & 1/77 \end{bmatrix}}$$

b. $(A^T)^{-1} = (A^{-1})^T$

$$= \boxed{\begin{bmatrix} -2/7 & 3/7 \\ 1/7 & 2/7 \end{bmatrix}}$$

c. $(7A)^{-1} = \frac{1}{7}A^{-1}$

$$= \boxed{\begin{bmatrix} -2/49 & 3/49 \\ 1/49 & 2/49 \end{bmatrix}}$$

THEOREM 2.10: CANCELLATION PROPERTIES

If C is an invertible matrix, then the following properties hold true.

1. If $AC = BC$ then $A = B$. Right cancellation property

2. If $CA = CB$ then $A = B$. Left cancellation property

THEOREM 2.11: SYSTEMS OF EQUATIONS WITH UNIQUE SOLUTIONS

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof:

Example 5: Use an inverse matrix to solve the system of equations.

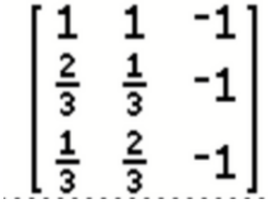
$$\begin{aligned} x_1 + \ \ x_2 - 2x_3 &= \ \ 0 \\ x_1 - 2x_2 + \ \ x_3 &= \ \ 0 \\ x_1 - \ \ x_2 - \ \ x_3 &= -1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$A^{-1} =$



$$\begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{2}{3} & -1 \end{bmatrix}$$

$$A^{-1}\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{x}$$

$$\boxed{x_1 = 1, x_2 = 1, x_3 = 1}$$

Section 2.4: ELEMENTARY MATRICES

When you are done with your homework you should be able to…

- π Factor a matrix into a product of elementary matrices
- π Find and use an LU-factorization of a matrix to solve a system of linear equations

DEFINITION OF AN ELEMENTARY MATRIX

An $n \times n$ matrix is called an elementary matrix when it can be obtained from the identity matrix I_n by a single elementary row operation.

Example 1: Identify the matrices that are elementary below, ~~show that B is the inverse of A~~.

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}$$

No - not square

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$B = -2R_2 + R_3$

yes

THEOREM 2.12: REPRESENTING ELEMENTARY ROW OPERATIONS

Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

Example 2: Let A, B, and C be

$$A=\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & 2\\ -1 & 2 & 0\end{bmatrix}\qquad B=\begin{bmatrix}-1 & 2 & 0\\ 0 & 1 & 2\\ 1 & 2 & -3\end{bmatrix}\qquad C=\begin{bmatrix}0 & 4 & -3\\ 0 & 1 & 2\\ -1 & 2 & 0\end{bmatrix}$$

Find an elementary matrix E such that $EA=C$.

$$\begin{bmatrix}e_{11} & e_{12} & e_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & 2\\ -1 & 2 & 0\end{bmatrix}=\begin{bmatrix}0 & 4 & -3\\ 0 & 1 & 2\\ -1 & 2 & 0\end{bmatrix}$$

$$c_{11} = e_{11}(1) + e_{12}(0) + e_{13}(-1) = 0$$
$$c_{12} = e_{11}(2) + e_{12}(1) + e_{13}(2) = 4$$
$$c_{13} = e_{11}(-3) + e_{12}(2) + e_{13}(0) = -3$$

$$e_{11} - e_{13} = 0 \rightarrow e_{11} = e_{13}$$
$$2e_{11} + e_{12} + 2e_{13} = 4 \rightarrow 2e_{11} + e_{12} + 2e_{11} = 4 \rightarrow 4e_{11} + e_{12} = 4$$
$$-3e_{11} + 2e_{12} = -3 \qquad 4e_{11} - 4 = e_{12}$$
$$\searrow$$
$$-3e_{11} + 2(4e_{11} - 4) = -3$$
$$-3e_{11} + 8e_{11} - 8 = -3$$
$$5e_{11} = 5$$
$$e_{11} = 1 = e_{13}$$
$$\text{and } e_{12} = 0$$

$$\boxed{E=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}$$

Example 3: Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form.

Matrix	Elem. Row op.	Elem. Matrix
$A=\begin{bmatrix}0&3&-3&6\\1&-1&2&-2\\0&0&2&2\end{bmatrix}$		
$\downarrow$		
$\begin{bmatrix}1&-1&2&-2\\0&3&-3&6\\0&0&2&2\end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$
$\downarrow$		
$\begin{bmatrix}1&-1&2&-2\\0&1&-1&2\\0&0&2&2\end{bmatrix}$	$\frac{1}{3}R_2 \rightarrow R_2$	$E_2=\begin{bmatrix}1&0&0\\0&\frac{1}{3}&0\\0&0&1\end{bmatrix}$
$\downarrow$		
$B=\begin{bmatrix}1&-1&2&-2\\0&1&-1&2\\0&0&1&1\end{bmatrix}$	$\frac{1}{2}R_3 \rightarrow R_3$	$E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/2\end{bmatrix}$

$$E_3E_2E_1A=B$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&3&-3&6\\1&-1&2&-2\\0&0&2&2\end{bmatrix}\overset{?}{=}\begin{bmatrix}1&-1&2&-2\\0&1&-1&2\\0&0&1&1\end{bmatrix}$$

Awesome!! ☺ It is true!

DEFINITION OF ROW EQUIVALENCE

Let A and B be $m\times n$ matrices. Matrix B is row-equivalent to A when there exists a finite number of elementary matrices, $E_1, E_2, \dots, E_k$ such that

$$B = E_k \cdot E_{k-1} \cdots E_2 \cdot E_1 \cdot A$$

THEOREM 2.13: ELEMENTARY MATRICES ARE INVERTIBLE

If E is an elementary matrix, then E^{-1} exists and is an invertible matrix.

Example 4: Find the inverse of the elementary matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

This is $-3R_2 + R_3 \to R_3$ from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E^{-1}E$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3R_2 + R_3 \to R_3 \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = E^{-1}$$

$$\begin{bmatrix} E & I_3 \end{bmatrix} \to \begin{bmatrix} I_3 & E^{-1} \end{bmatrix}$$

THEOREM 2.15: EQUIVALENT CONDITIONS

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\vec{b}$.
3. $A\mathbf{x} = O$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.

THE *LU*-FACTORIZATION

3x3 lower triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3x3 upper triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

DEFINITION OF *LU*-FACTORIZATION

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an ***LU*-factorization** of A.

Example 5: Solve the linear system $A\mathbf{x}=\mathbf{b}$ by

a. Finding an LU-factorization of the coefficient matrix A.
b. Solving the lower triangular system $L\mathbf{y}=\mathbf{b}$.
c. Solving the upper triangular system $U\mathbf{x}=\mathbf{y}$.

$$\begin{aligned} 2x_1 \qquad\qquad &= 4 \\ -2x_1 + \ x_2 - x_3 \qquad &= -4 \\ 6x_1 + 2x_2 + x_3 \qquad &= 15 \\ -x_4 &= -1 \end{aligned}$$

a) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\downarrow$

$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ $\quad R_2 + R_3 \to R_2 \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$L = E_1 A \longrightarrow E_1^{-1} L = E_1^{-1} E_1 A \longrightarrow A = E_1^{-1} L$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

E_1^{-1} can be obtained by applying $R_2 + (-R_3) \to R_2$ to I_4

$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$

and $A = UL$

So we didn't have to do all that other work!

** Please note that we found a UL Factorization not an LU.

Section 2.5: APPLICATIONS OF MATRIX OPERATIONS

When you are done with your homework you should be able to…

- π Write and use a stochastic matrix
- π Use matrix multiplication to encode and decode messages

STOCHASTIC MATRICES

Many types of applications involve a finite set of ______________________ ___________________________ of a given population.

The ______________________ that a member of a population will change from the ____________ state to the ____________ state is represented by a number ___________, where _____________________. A probability of ______________ means that the member is certain ____________ to change from the *j*th state to the *i*th state whereas A probability of ______________ means that the member is ____________ to change from the *j*th state to the *i*th state.

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

***P* is called the ______________ of ________________ probabilities.** At each transition, each member in a given state must either stay in that state or change to another state. Therefore, the sum of the entries in any ________________ is _____. This type of matrix is called ____________________. An ___________ matrix *P* is a **stochastic matrix** when each entry is a number between ______ and ______ inclusive.

Example 1: Determine whether the matrix is stochastic.

$$A=\begin{bmatrix}0.35 & 0.2\\ 0.65 & 0.75\end{bmatrix}$$

$$B=\begin{bmatrix}\frac{1}{8} & \frac{3}{5} & \frac{1}{12}\\ \frac{1}{2} & \frac{1}{10} & \frac{1}{3}\\ \frac{3}{8} & \frac{3}{10} & \frac{7}{12}\end{bmatrix}$$

Example 2: A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% probability that an infected mouse will overcome the virus, and during the same week, there is a 10% probability that a noninfected will become infected. One hundred mice are currently infected with the virus. How many will be infected (a) next week and (b) in two weeks?

CRYPTOGRAPHY

A _______________________ is a message written according to a secret code. Suppose we assign a number to each letter in the alphabet.

0	_	14	N
1	A	15	O
2	B	16	P
3	C	17	Q
4	D	18	R
5	E	19	S
6	F	20	T
7	G	21	U
8	H	22	V
9	I	23	W
10	J	24	X
11	K	25	Y
12	L	26	Z
13	M		

Example 3: Write the uncoded row matrices of size 1 x 3 for the message TARGET IS HOME.

Example 4: Use the following invertible matrix to encode the message TARGET IS HOME.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

Section 3.1: THE DETERMINANT OF A MATRIX

When you are done with your homework you should be able to…

- π Find the determinant of a 2 x 2 matrix
- π Find the minors and cofactors of a matrix
- π Use expansion by cofactors to find the determinant of a matrix
- π Find the determinant of a triangular matrix

Every square matrix can be associated with a real number called its determinant. Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations.

Consider the system

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{b_1 - a_{12}x_2}{a_{11}}$$

$$a_{21}\left(\frac{b_1 - a_{12}x_2}{a_{11}}\right) + a_{22}x_2 = b_2$$

$$\frac{a_{21}b_1 - a_{12}a_{21}x_2 + a_{11}a_{22}x_2}{a_{11}} = b_2$$

$$a_{21}b_1 + x_2(a_{11}a_{22} - a_{12}a_{21}) = a_{11}b_2$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}},$$

$$x_1 = \left[b_1 - a_{12}\frac{(a_{11}b_2 - a_{21}b_1)}{a_{11}a_{22} - a_{12}a_{21}}\right] \cdot \frac{1}{a_{11}}$$

$$x_1 = \frac{b_1(a_{11}a_{22} - a_{12}a_{21}) - a_{11}a_{12}b_2 + a_{12}a_{21}b_1}{a_{11}(a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{a_{11}a_{22}b_1 - a_{12}\cancel{a_{21}}b_1 - a_{11}a_{12}b_2 + a_{12}\cancel{a_{21}}b_1}{a_{11}(a_{11}a_{22} - a_{12}a_{21})}$$

$$x_1 = \frac{\overset{1}{\cancel{a_{11}}}(a_{22}b_1 - a_{12}b_2)}{\underset{1}{\cancel{a_{11}}}(a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

So, we have found that

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}} \quad \text{and} \quad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{21} a_{12}}$$

DEFINITION OF THE DETERMINANT OF A 2 x 2 MATRIX

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$.

**Note: In this text, det (A) and |A| are used interchangeably to represent the determinant of a matrix. In this context, the vertical bars are used to represent the determinant of a matrix as opposed to the absolute value.

Example 1: Find $|A|$ and $|B|$.

$$A = \begin{bmatrix} -1 & 4 \\ 11 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 21 & -3 \\ -6 & 10 \end{bmatrix}$$

$$\det(A) = (-1)(7) - (11)(4)$$
$$= \boxed{-51}$$

$$\det(B) = (21)(10) - (-6)(-3)$$
$$= \boxed{192}$$

DEFINITION OF MINORS AND COFACTORS OF A MATRIX

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Example 2: Find the minor and cofactor of a_{12} .and b_{23} .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$M_{23} = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} = 2(-2) - (-1)(3) = \boxed{-1}$$

$$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -(-1) = \boxed{1}$$

Consider:

$$A=\begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{bmatrix} \qquad B=\begin{bmatrix} \overset{+}{b_{11}} & \overset{-}{b_{12}} & \overset{+}{b_{13}} & \overset{-}{b_{14}} \\ \overset{-}{b_{21}} & \overset{+}{b_{22}} & \overset{-}{b_{23}} & \overset{+}{b_{24}} \\ \overset{+}{b_{31}} & \overset{-}{b_{32}} & \overset{+}{b_{33}} & \overset{-}{b_{34}} \\ \overset{-}{b_{41}} & \overset{+}{b_{42}} & \overset{-}{b_{43}} & \overset{+}{b_{44}} \end{bmatrix}$$

What do you notice?

DEFINITION OF THE DETERMINANT OF A SQUARE MATRIX

If A is a square matrix of order $n>2$, then the determinant of ~~the~~ A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,

$$\det(A)=|A|=\sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11}+a_{12}C_{12}+\cdots+a_{1n}C_{1n}.$$

$\det(A)=\sum_{i=1}^{n} a_{ij}C_{ij}$ → jth column expansion

$\det(A)=\sum_{j=1}^{n} a_{ij}C_{ij}$ → ith row expansion

* It can be any row or column

Example 3: Confirm that, for 2x2 matrices, this definition yields $|A| = a_{11}a_{22} - a_{21}a_{12}$.

$$A=\begin{bmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}) + (-1)a_{12}(a_{21})$$
$$= a_{11}a_{22} - a_{21}a_{12}$$

Example 4: Find $|B|$.

$$B = \begin{bmatrix} \overset{+}{2} & -1 & 4 \\ \overset{-}{0} & 1 & 3 \\ \overset{+}{3} & -2 & 1 \end{bmatrix}$$

$$\det(B) = 2\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} - 0 + 3\begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 2(1-(-6)) + 3(-3-4)$$

$$= 14 - 21$$

$$= \boxed{-7}$$

Expansion by Cofactors

THEOREM 3.1: ~~DETERMINANT OF A MATRIX PRODUCT~~

If A be a square matrix of order n. Then the determinant of A is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$ (*i*th row expansion)

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{ij}C_{ij} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$ (*j*th ~~column~~ column expansion)

Is there an easier way to complete the previous example?

Example 4: Find $|B|$.

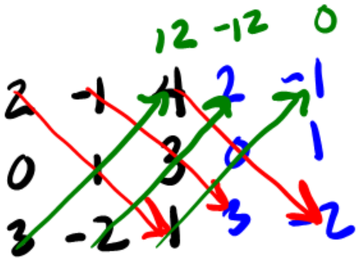
$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

use the row or column which has the most zeros as your expansion row or column.

Alternative Method to evaluate the determinant of a 3x3 matrix: Copy the first and second columns of the matrix to form fourth and fifth columns. Then obtain the determinant by adding (or subtracting) the products of the six diagonals.

Example 4: Find $|B|$.

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$



sum: 2, -9, 0

$$|B| = [2 + (-9) + 0] - [12 + (-12) + 0] = \boxed{-7}$$

Example 5: Find $\det(A)$.

$$A = \begin{array}{cccc} + & - & + & - \\ \hline 1 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ 6 & -1 & 2 & 0 \\ -3 & 5 & -8 & 7 \end{array}$$

$$\det(A) = 1 \begin{array}{|ccc|} + & - & + \\ 7 & 0 & 0 \\ -1 & 2 & 0 \\ 5 & -8 & 7 \end{array} - 0 + 0 - 0$$

$$\det(A) = 7 \begin{vmatrix} 2 & 0 \\ -8 & 7 \end{vmatrix} - 0 + 0$$

$$\det(A) = 7(14-0)$$

$$\boxed{\det(A) = 98}$$

What did you notice?

We like zeros and hey... the product of the main diagonal entries is also 98.

THEOREM 3.2: DETERMINANT OF A TRIANGULAR MATRIX

If A is a triangular matrix of order n, then its determinant is the

___product___ of the ___entries___ on the ___main___

___diagonal___. That is,

$\det(A) = |A| =$ ___$a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$___.

Example 6: Find the values of $\overset{\nearrow\text{lamda}}{\lambda}$, for which the determinant is zero.

$$\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3)-4=0$$

$$\lambda^2-4\lambda+3-4=0$$

$$\lambda^2-4\lambda-1=0$$

$$\lambda=\frac{4\pm\sqrt{16-(-4)}}{2}$$

$$\lambda=\frac{4\pm2\sqrt{5}}{2}$$

$$\boxed{\lambda=2\pm\sqrt{5}}$$

Section 3.2: DETERMINANTS AND ELEMENTARY OPERATIONS

When you are done with your homework you should be able to…

- π Use elementary row operations to evaluate a determinant
- π Use elementary column operations to evaluate a determinant
- π Recognize conditions that yield zero determinants

Consider the following two determinants:

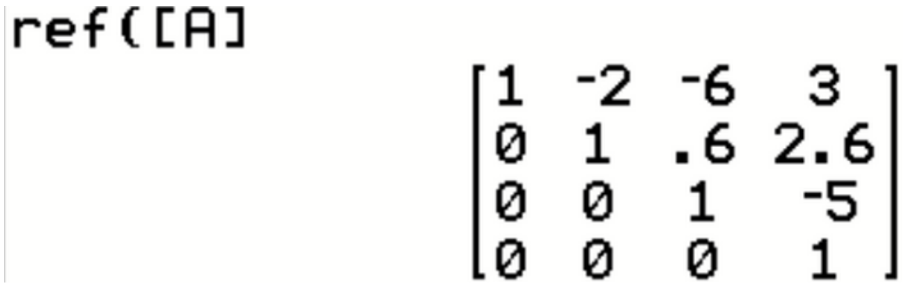
$$A=\begin{bmatrix}-1 & 2 & 6 & -2\\ -2 & 9 & 15 & 7\\ 3 & -6 & -17 & 4\\ -5 & 10 & 30 & -15\end{bmatrix} \qquad B=\begin{bmatrix}-1 & 2 & 6 & -2\\ 0 & 5 & 3 & 11\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & -5\end{bmatrix}$$

Find the determinant of each matrix.

$|A| = \boxed{25}$

$$\text{ref}(B)=\begin{bmatrix}1 & -2 & -6 & 2\\ 0 & 1 & 3/5 & 11/5\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$|B| = (-1)(5)(1)(-5) = \boxed{25}$



```
ref([A]
              [1  -2  -6   3  ]
              [0   1  .6  2.6 ]
              [0   0   1  -5  ]
              [0   0   0   1  ]
```

hmm... evil plan didn't quite work!

$$\text{rref}(A)=\text{rref}(B)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

What did you find out?

$|A| = |B|$

Take a closer look at the two matrices. Do you notice anything?

ops on B:

$R_2 = 2R_1 + R_2 \rightarrow$ results in R_2 from A

$R_3 = -3R_1 + R_3 \rightarrow$ " " R_3 " "

$R_4 = 5R_1 + R_4 \rightarrow$ " " R_4 " "

THEOREM 3.3: ELEMENTARY ROW OPERATIONS AND DETERMINANTS

Let A and B be square matrices.

1. When B is obtained from A by interchanging two rows of A, $\det(B) = -\det(A)$.
2. When B is obtained from A by adding a multiple of a row of A to another row of A, $\det(B) = \det(A)$.
3. When B is obtained from A by multiplying a row of A by a nonzero constant c, $\det(B) = c\det(A)$.

NOTE: Theorem 3.3 remains valid when the word “column” replaces the word “row”. Operations performed on columns are called elementary column operations.

Example 1: Determine which property of determinants the equation illustrates.

a.

$$A = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 12 & 7 \\ 3 & -3 & 8 \end{vmatrix} = -\begin{vmatrix} 3 & -1 & 1 \\ 7 & 12 & 4 \\ 8 & -3 & 3 \end{vmatrix} = B$$

#1 $\det(B) = -\det(A)$

b.

$$A = \begin{vmatrix} 2 & -4 & 2 \\ 6 & 10 & 2 \\ 8 & -4 & 6 \end{vmatrix} = 8\begin{vmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & -2 & 3 \end{vmatrix} = B$$

$$\det(A) = \underset{\underset{2}{\downarrow}}{c_1}\,\underset{\underset{2}{\downarrow}}{c_2}\,\underset{\underset{2}{\downarrow}}{c_3}\,\det(B)$$

#3

Example 2: Use elementary row or column operations to find the determinant.

$$\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 8 \end{vmatrix}$$

$$= (3)(-5)(8)$$

$$= \boxed{-120}$$

$$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{bmatrix}$$

$$\downarrow -2R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -15 & 20 \end{bmatrix}$$

$$\downarrow -3R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

THEOREM 3.4: CONDITIONS THAT YIELD A ZERO DETERMINANT

If A is a square matrix, and any one of the following conditions is true, then $\det(A) = 0$.

1. An entire row (or column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

	Cofactor Expansion		Row Reduction	
Order n	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

Example 3: Prove the property.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\ a \neq 0,\ b \neq 0,\ c \neq 0.$$

Proof: Let a, b, c $\in \mathbb{R}$ and nonzero.

$$\begin{vmatrix} \overset{+}{1+a} & 1 & 1 \\ \overset{-}{1} & 1+b & 1 \\ \overset{+}{1} & 1 & 1+c \end{vmatrix} = (1+a)\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1+b & 1 \end{vmatrix}$$

$$= (1+a)\left[(1+b)(1+c)-1\right] - \left[1+c-1\right] + \left[1-(1+b)\right]$$

$$= (1+a)(\cancel{1}+b+c+bc-\cancel{1}) - c - b$$

$$= \cancel{b}+\cancel{c}+bc+ab+ac+abc-\cancel{c}-\cancel{b}$$

$$= abc\left(\frac{1}{a} + \frac{1}{c} + \frac{1}{b} + 1\right)$$

$$= abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \; //$$

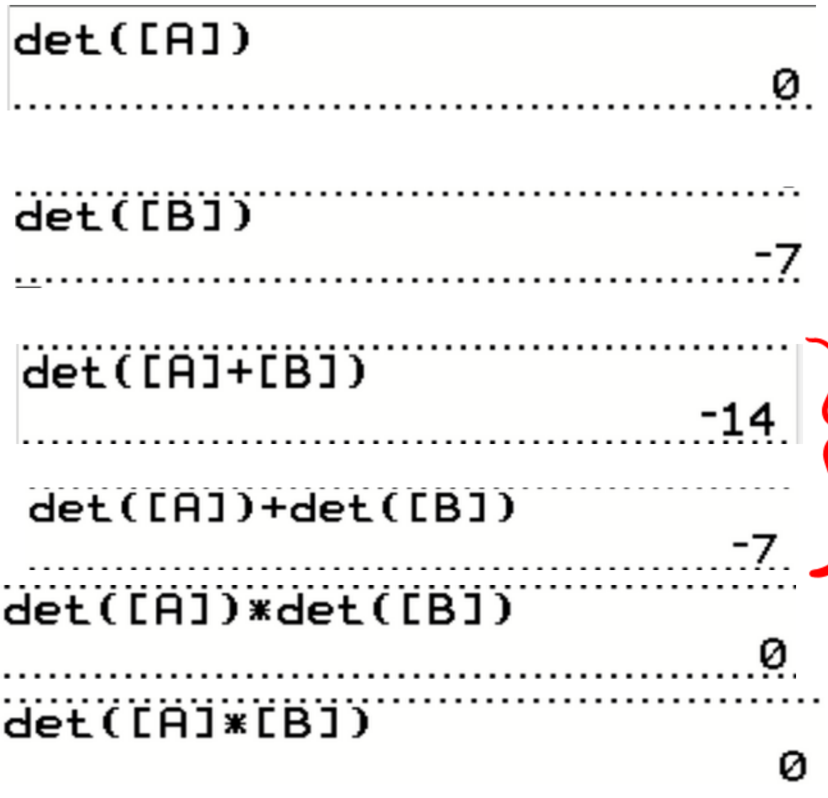
Section 3.3: PROPERTIES OF DETERMINANTS

When you are done with your homework you should be able to…

- Find the determinant of a matrix product and a scalar multiple of a matrix
- Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix
- Find the determinant of the transpose of a matrix

Example 1: Find $|A|$, $|B|$, $|A||B|$, $|A+B|$, $|A|+|B|$ and $|AB|$.

$$A=\begin{bmatrix}2 & 0 & 1\\ 1 & -1 & 2\\ 3 & 1 & 0\end{bmatrix} \qquad B=\begin{bmatrix}2 & -1 & 4\\ 0 & 1 & 3\\ 3 & -2 & 1\end{bmatrix}$$



```
det([A])
                                0
det([B])
                               -7
det([A]+[B])
                              -14
det([A])+det([B])
                               -7
det([A])*det([B])
                                0
det([A]*[B])
                                0
```

clearly…
det (A+B) ≠ det (A)+det (B)

hmm…
maybe det (AB) = det (A)det(B)

THEOREM 3.5: DETERMINANT OF A MATRIX PRODUCT

If A and B are square matrices of order n, then

$$\det(AB) = \det(A)\det(B)$$

Example 2: Find $|3A|$ and $|3B|$.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$\det A = 10-(-3) = 13$

$$\det(3A) = \begin{vmatrix} 3 & -3 \\ 9 & 30 \end{vmatrix}$$

$= 90-(-27)$

$= 117$

$= 9 \cdot 13 \longrightarrow = 3^2 \cdot 13$

$\det(B) = -7$

$\det(3B) = -189 \overset{?}{=} 3^3 \cdot (-7) = -189$

whoo hoo!!

So maybe $\det(cA) = c^n \det(A)$...

THEOREM 3.6: DETERMINANT OF A SCALAR MULTIPLE OF A MATRIX

If A is a square matrix of order n and c is a scalar, then the determinant of $|cA|$ is

$c^n \det(A)$.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$, $c, a_{ij} \in \mathbb{R}$.

$\text{Det}(B) = ca_{11}ca_{22} - ca_{21}ca_{12}$

$= c^2 a_{11}a_{22} - c^2 a_{21}a_{12}$

$= c^2(a_{11}a_{22} - a_{21}a_{12})$

$= c^2 \det(A).$ //

Example 3: Find A^{-1}, $|A|$, B^{-1}, and $|B|$.

$$A = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 2 \\ 11 & 7 \end{bmatrix}$$

$$[A\ \ I_2] = \left[\begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$-2R_1 + 3R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} -3 & 6 & 1 & 0 \\ 0 & 0 & -2 & 3 \end{array}\right]$$

impossible to get I_2

A is singular.

$$\det(A) = (-3)(4) - (6)(-2)$$
$$= 0$$

$$[B\ \ I_2] = \left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 11 & 7 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$-11R_1 + 5R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 0 & 13 & -11 & 5 \end{array}\right]$$

$$\downarrow$$

$$-13R_1 + 2R_2 \to R_1$$

$$\left[\begin{array}{cc|cc} -65 & 0 & -35 & 10 \\ 0 & 13 & -11 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7/13 & -2/13 \\ 0 & 1 & -11/13 & 5/13 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 7/13 & -2/13 \\ -11/13 & 5/13 \end{bmatrix}$$

$$\det(B) = (5)(7) - (2)(11)$$
$$= 13$$

$$B^{-1} = \frac{1}{13}\begin{bmatrix} 7 & -2 \\ -11 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ 11 & 7 \end{bmatrix}$$

so in general, if A is 2x2 and $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

THEOREM 3.7: DETERMINANT OF AN INVERTIBLE MATRIX

A square matrix A is invertible (nonsingular) if and only if

$$\det(A) \neq 0$$

Example 4: Find $|A|$ and $|A^{-1}|$.

$$A = \begin{bmatrix} -3 & 3 \\ -2 & 1 \end{bmatrix}$$

$$\boxed{|A| = -3-(-6) = 3}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 1 & -3 \\ 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -3 \\ 2 & -3 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1/3 & -1 \\ 2/3 & -1 \end{bmatrix}}$$

$$\boxed{|A^{-1}| = -1/3 - (-2/3) = 1/3}$$

THEOREM 3.8: DETERMINANT OF AN INVERSE MATRIX

If A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof:

Since A is invertible, $AA^{-1} = I$ and $|A||A^{-1}| = |I| = 1$.

Since A is invertible, $\det(A) \neq 0$. $\therefore |A||A^{-1}| = 1$

$$|A^{-1}| = \frac{1}{|A|} \quad //$$

EQUIVALENT CONDITIONS FOR A NONSINGULAR MATRIX

If A is an $n \times n$ nonsingular matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix. $\vec{b}$
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$.

Example 5: Determine if the system of linear equations has a unique solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 - x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{bmatrix} \end{array}$$

$$\det(A) = 1\begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}$$

$$= -2-(2)-(4-(-3))-(-4-(-3))$$

$$= -2-2\ -7-(-1)$$

$$= -10 \neq 0$$

$\therefore$ the system has a unique solution.

Example 6: Find $|A|$ and $|A^T|$.

$$A = \begin{bmatrix} 7 & 12 \\ 2 & -2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & 2 \\ 12 & -2 \end{bmatrix} \qquad \det(A^T) = -14-24 = -38$$

$$\det(A) = -14-24 = -38$$

THEOREM 3.9: DETERMINANT OF A TRANSPOSE

If A is a square matrix, then

$$\det(A) = \det(A^T)$$

Section 3.4: APPLICATIONS OF DETERMINANTS

When you are done with your homework you should be able to…

- π Use Cramer's Rule to solve a system of *n* linear equations
- π Use determinants to find area, volume, and the equations of lines and planes

Example 1: Solve the system of linear equations. Assume that $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$|A| = a_{11}a_{22} - a_{21}a_{12}$

$x_1 = \dfrac{b_1 - a_{12}x_2}{a_{11}}$

$a_{21}x_1 + a_{22}x_2 = b_2$

$a_{21}\left(\dfrac{b_1 - a_{12}x_2}{a_{11}}\right) + a_{22}x_2 = b_2$

$\dfrac{b_1a_{21} - a_{21}a_{12}x_2 + a_{11}a_{22}x_2}{a_{11}} = b_2$

$b_1a_{21} - a_{21}a_{12}x_2 + a_{11}a_{22}x_2 = b_2a_{11}$

$(a_{11}a_{22} - a_{21}a_{12})x_2 = b_2a_{11} - b_1a_{21}$

$x_2 = \dfrac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{21}a_{12}} = \boxed{\dfrac{b_2a_{11} - b_1a_{21}}{|A|}}$

$x_1 = \dfrac{b_1 - a_{12}x_2}{a_{11}}$

$x_1 = \left[b_1 - a_{12}\left(\dfrac{b_2a_{11} - b_1a_{21}}{|A|}\right)\right] \cdot \dfrac{1}{a_{11}}$

$x_1 = \dfrac{b_1}{a_{11}} - \dfrac{b_2a_{12}a_{11} - b_1a_{12}a_{21}}{a_{11}|A|}$

$$x_1 = \frac{|A|\, b_1 - b_2 a_{12} a_{11} + b_1 a_{12} a_{21}}{a_{11}\, |A|}$$

$$x_1 = \frac{(a_{11} a_{22} - a_{21} a_{12}) b_1 - b_2 a_{12} a_{11} + b_1 a_{12} a_{21}}{a_{11}\, |A|}$$

$$x_1 = \frac{b_1 a_{11} a_{22} - \cancel{b_1 a_{21} a_{12}} - b_2 a_{12} a_{11} + \cancel{b_1 a_{12} a_{21}}}{a_{11}\, |A|}$$

$$x_1 = \frac{\overset{1}{\cancel{a_{11}}} (b_1 a_{22} - b_2 a_{12})}{\underset{1}{\cancel{a_{11}}}\, |A|}$$

$$\boxed{x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{|A|}}$$

consider:

$$b_1 a_{22} - b_2 a_{12} = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

For A_i, the ith column of the coefficient matrix A is replace with the column vector $\vec{b}$.

THEOREM 3.11: CRAMER'S RULE

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)}$$

Where the ith column of A_i is the column of constants in the system of equations.

Proof: The adjoint of a matrix is the transpose of the matrix of cofactors. $\det(A) = |A| = \sum_{j=1}^{n} a_{ij} C_{ij}$ (ith row expansion). $\det(A) = |A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ (jth column expansion). Let the system be represented by $A\vec{x} = B$. $\vec{x} = A^{-1}B = \frac{1}{|A|}\operatorname{adj}(A)B$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Entries of B are $b_1, b_2, \ldots, b_n$, so

$$x_i = \frac{1}{|A|}\left(b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}\right)$$

(Cofactor expansion of A).

$$\therefore x_i = \frac{|A_i|}{|A|}.$$

Example 2: If possible, use Cramer's Rule to solve the system.

a.

$$-x_1 - 2x_2 = 7$$
$$2x_1 + 4x_2 = 11$$

$A = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$, $\det(A) = 0$, so we can't use Cramer's Rule.

These are parallel lines, so there's no solution.

$\boxed{\emptyset, \text{inconsistent}}$

b.

$$\begin{aligned} -8x_1 + 7x_2 - 10x_3 &= -151 \\ 12x_1 + 3x_2 \quad - 5x_3 &= \quad 86 \\ 15x_1 - 9x_2 \quad + 2x_3 &= \quad 187 \end{aligned}$$

$$A = \begin{bmatrix} -8 & 7 & -10 \\ 12 & 3 & -5 \\ 15 & -9 & 2 \end{bmatrix}$$

$|A| = 1149 \neq 0$ $\exists$ a unique solution and we can use Cramer's Rule.

$$|A_1| = \begin{vmatrix} -151 & 7 & -10 \\ 86 & 3 & -5 \\ 187 & -9 & 2 \end{vmatrix} = 11490 \qquad x_1 = \frac{11490}{1149} = 10$$

$$|A_2| = \begin{vmatrix} -8 & -151 & -10 \\ 12 & 86 & -5 \\ 15 & 187 & 2 \end{vmatrix} = -3447 \qquad x_2 = \frac{-3447}{1149} = -3$$

$$|A_3| = \begin{vmatrix} -8 & 7 & -151 \\ 12 & 3 & 86 \\ 15 & -9 & 187 \end{vmatrix} = 5745 \qquad x_3 = \frac{5745}{1149} = 5$$

$$\boxed{\{(10, -3, 5)\}}$$

AREA OF A TRIANGLE IN THE *xy*-PLANE

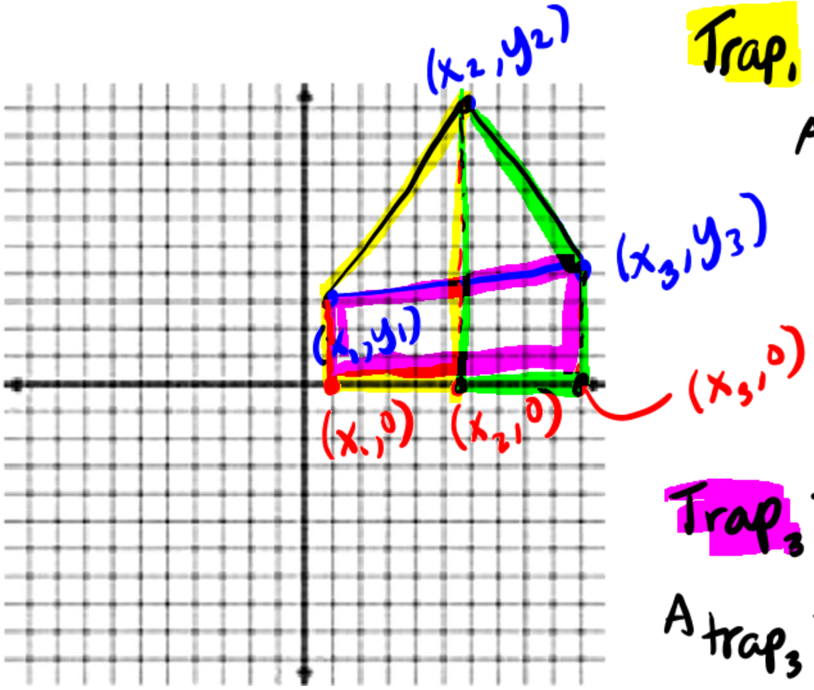
The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign ($\pm$) is chosen to give positive area.

Proof:

$$A_{trap} = \left(\frac{b_1+b_2}{2}\right)\cdot h$$



$Trap_1$: $(x_1, 0), (x_1, y_1), (x_2, 0), (x_2, y_2)$

$$A_{trap_1} = \frac{y_1+y_2}{2}\cdot(x_2 - x_1)$$

$Trap_2$: $(x_2, 0), (x_2, y_2), (x_3, 0), (x_3, y_3)$

$$A_{trap_2} = \frac{y_2+y_3}{2}\cdot(x_3 - x_2)$$

$Trap_3$: $(x_1, 0), (x_1, y_1), (x_3, 0), (x_3, y_3)$

$$A_{trap_3} = \frac{y_1+y_3}{2}(x_3 - x_1)$$

$$A_\Delta = \frac{1}{2}\left[(y_1+y_2)\cdot(x_2-x_1) + (y_2+y_3)\cdot(x_3-x_2) - (y_1+y_3)(x_3-x_1)\right]$$

Example 3: Find the area of the triangle whose vertices are $(1,-1)$, $(3,-5)$, and $(0,-2)$.

$$A = \pm\frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 3 & -5 & 1 \\ 0 & -2 & 1 \end{vmatrix}$$

$$A = \pm\frac{1}{2}(-6)$$

$$\boxed{A = 3 \text{ sq. units}}$$

TEST FOR COLLINEAR POINTS IN THE *xy*-PLANE

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

TWO-POINT FORM OF THE EQUATION OF A LINE

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

Example 4: Find an equation of the line passing through the points $(-4,7)$ and $(2,4)$.

$$\begin{vmatrix} x & y & \overset{+}{1} \\ -4 & 7 & \overset{-}{1} \\ 2 & 4 & \overset{+}{1} \end{vmatrix} = 0$$

$$1\begin{vmatrix} -4 & 7 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} x & y \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} x & y \\ -4 & 7 \end{vmatrix} = 0$$

$$(-16-14)-(4x-2y)+(7x-4y)=0$$

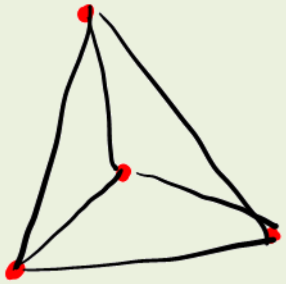
$$-30-4x+2y+7x-4y=0$$

$$\boxed{3x-2y-30 \quad =0}$$

VOLUME OF A TETRAHEDRON

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$V = \pm \frac{1}{6} \det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$



where the sign ($\pm$) is chosen to give positive volume.

Example 5: Find the volume of the tetrahedron with vertices $(1,1,1)$, $(0,0,0)$, $(2,1,-1)$, and $(-1,1,2)$.

$$V = \pm \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 1 \end{vmatrix} = \frac{1}{2} \text{ cubic units}$$

TEST FOR COPLANAR POINTS IN SPACE

Four points, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$$

~~where the sign (±) is chosen to give positive volume.~~

THREE-POINT FORM OF THE EQUATION OF A ~~LINE~~ PLANE

An equation of the plane passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2) ,and (x_3, y_3, z_3) is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

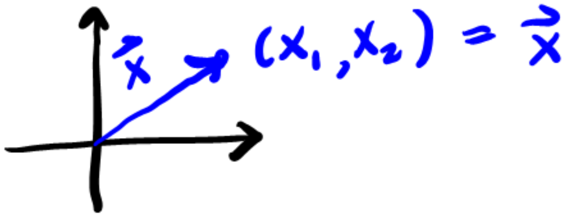
Section 4.1: VECTORS IN R^n

When you are done with your homework you should be able to…

- Represent a vector as a directed line segment
- Perform basic vector operations in R^2 and represent them graphically
- Perform basic vector operations in R^n
- Write a vector as a linear combination of other vectors

VECTORS IN THE PLANE

A vector is characterized by two quantities, length and direction, and is represented by a directed line segment. Geometrically, a vector in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2).

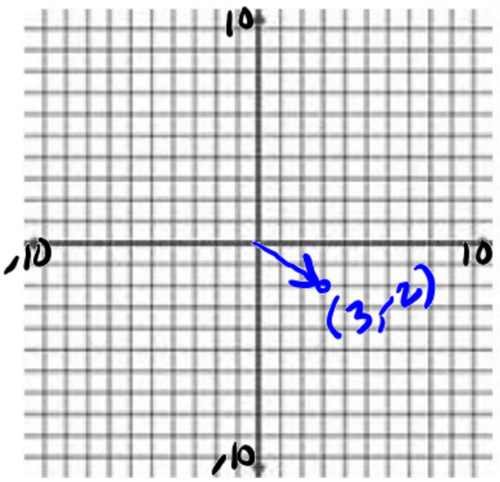


The same ordered pair used to represent its terminal point also represents the vector. That is, $\vec{x} = (x_1, x_2)$.

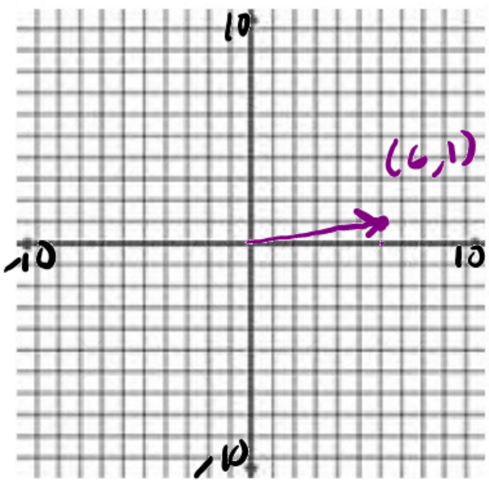
The coordinates x_1 and x_2 are called the components of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$.

Example 1: Use a directed line segment to represent the vector, and give the graphical representation of the vector operations.

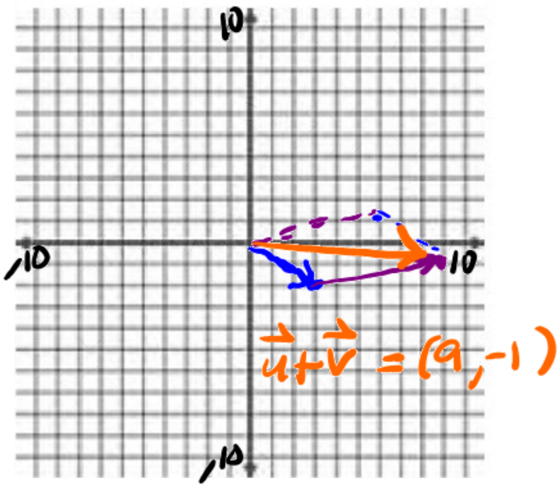
a. $\mathbf{u} = (3, -2)$



b. $\mathbf{v} = (6, 1)$

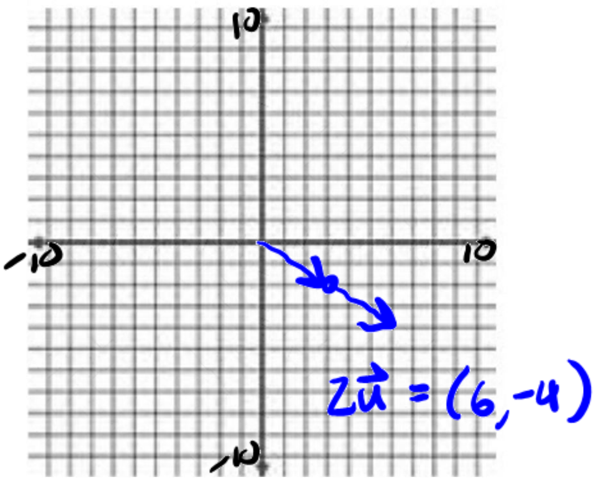


c. $\mathbf{u} + \mathbf{v}$



$$\vec{u} + \vec{v} = (3,-2) + (6,1)$$
$$= (3+6, -2+1)$$
$$= (9,-1)$$

d. $2\mathbf{u}$



$$2\vec{u} = 2(3,-2)$$
$$= (6,-4)$$

THEOREM 4.1: PROPERTIES OF VECTOR ADDITION AND SCALAR MULTIPLICATION IN THE PLANE

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u}+\mathbf{v}$ is a vector in the plane. closure under addition

2. $\mathbf{u}+\mathbf{v} = \vec{v}+\vec{u}$ commutative property of addition

   Proof: Let $\vec{u}=(u_1,u_2)$, $\vec{v}=(v_1,v_2)$, $u_i, v_i \in \mathbb{R}$.

   $\vec{u}+\vec{v} = (u_1,u_2)+(v_1,v_2)$

   $= (u_1+v_1, u_2+v_2)$

   $= (v_1+u_1, v_2+u_2)$

   $= (v_1,v_2)+(u_1,u_2)$

   $= \vec{v}+\vec{u}$ //

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\vec{u}+\vec{v})+\vec{w}$ associative property of addition

4. $\mathbf{u}+\mathbf{0} = \vec{u}$ additive identity property

5. $\mathbf{u}+(-\mathbf{u}) = \vec{0}$ additive inverse property

6. $c\mathbf{u}$ is a vector in the plane. closure under scalar mult.

   Proof: Let $\vec{u}=(u_1,u_2)$, $c, u_i \in \mathbb{R}$.

   $c\vec{u} = c(u_1,u_2)$

   $= (cu_1, cu_2)$; which is a vector in the plane. //

7. $c(\mathbf{u}+\mathbf{v}) = c\vec{u}+c\vec{v}$ Distributive property

8. $(c+d)\mathbf{u} = c\vec{u}+d\vec{u}$ Distributive property

9. $c(d\mathbf{u}) = (cd)\vec{u}$ associative property of scalar multiplication

10. $1(\vec{u}) = \vec{u}$ multiplicative identity property

10. $1(\mathbf{u}) =$ See above page property

Example 2: Let $\mathbf{u} = (-1,1,4)$, $\mathbf{v} = (0,3,-3)$, and $\mathbf{w} = (7,5,-1)$

a. $\mathbf{u} + 6\mathbf{v}$

$= (-1,1,4) + 6(0,3,-3)$

$= (-1,1,4) + (0,18,-15)$

$= (-1+0, 1+18, 4+(-18))$

$= \boxed{(-1, 19, -14)}$

b. $-\mathbf{v} + 2(\mathbf{u} + \mathbf{w})$

$= -(0,3,-3) + 2[(-1,1,4) + (7,5,-1)]$

$= (0,-3,3) + 2(6,6,3)$

$= (0,-3,3) + (12,12,6)$

$= \boxed{(12,9,9)}$

THEOREM 4.2: PROPERTIES OF VECTOR ADDITION AND SCALAR MULTIPLICATION IN R^n

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u}+\mathbf{v}$ is a vector in R^n. closure under addition
2. $\mathbf{u}+\mathbf{v} = \vec{v}+\vec{u}$ commutative property of addition
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\vec{u}+\vec{v})+\vec{w}$ Associative property of addition
4. $\mathbf{u}+\mathbf{0} = \vec{u}$ additive identity property
5. $\mathbf{u}+(-\mathbf{u}) = \vec{0}$ additive inverse property
6. $c\mathbf{u}$ is a vector in R^n. closure under scalar mult.
7. $c(\mathbf{u}+\mathbf{v}) = c\vec{u}+c\vec{v}$ Distributive property
8. $(c+d)\mathbf{u} = c\vec{u}+d\vec{u}$ Distributive property
9. $c(d\mathbf{u}) = (cd)\vec{u}$ Associative property of multiplication

10. $1(\vec{u}) = \vec{u}$ Mult. identity property

10. $1(\mathbf{u}) =$ ____ See above ____________________ ________________property

IMPORTANT VECTOR SPACES

R^1 = 1-space = the set of real numbers.

R^2 = 2-space = the set of all ordered pairs of real numbers.

R^3 = 3-space = the set of all ordered triples of real numbers.

R^n = n-space = the set of all ordered n-tuples of real numbers.

Let $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in R^n, and let $c \in \mathbb{R}$. Then the sum of $\vec{u}$ and $\vec{v}$ is defined as the vector, $\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$ and the scalar multiplication of $\vec{u}$ by c is defined as the vector, $c\vec{u} = (cu_1, cu_2, \ldots, cu_n)$.

Example 3: Let $\mathbf{u} = (0,4,3,4,4)$ and $\mathbf{v} = (6,8,-3,3,-5)$.

a. $\mathbf{u} - \mathbf{v} = \vec{u} + (-\vec{v})$

$= (-6,-4,6,1,9)$

b. $4(\mathbf{u} + 3\mathbf{v})$

$= 4\vec{u} + 12\vec{v}$

$= (0,16,12,16,16) +$

$(72,96,-36,36,-60)$

$= \boxed{(72,112,-24,52,-44)}$

THEOREM 4.3: PROPERTIES OF ADDITIVE IDENTITY AND ADDITIVE INVERSE

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the following properties are true.

1. The additive identity is unique.

   Proof: Assume $\vec{v}+\vec{u}=\vec{v}$.

   $(\vec{v}+\vec{u})+(-\vec{v})=\vec{v}+(-\vec{v})$

   $(\vec{u}+\vec{v})+(-\vec{v})=\vec{0}$

   $\vec{u}+(\vec{v}+(-\vec{v}))=\vec{0}$

   $\vec{u}+\vec{0}=\vec{0}$

   $\vec{u}=\vec{0}$. //

2. The additive inverse is unique.
3. $0\mathbf{v}=$ $\vec{0}$
4. $c\mathbf{0}=$ $\vec{0}$
5. If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\vec{v}=\vec{0}$.
6. $-(-\mathbf{v})=$ $\vec{v}$

Example 4: Solve for $\mathbf{w}$, where $\mathbf{u}=(2,-1,3,4)$ and $\mathbf{v}=(-1,8,0,3)$.

a. $\mathbf{w}+\mathbf{u}=-\mathbf{v}$

$$(w+\vec{u})+(-\vec{u})=-\vec{v}\quad+(-\vec{u})$$

$$\vec{w}+(\vec{u}+(-\vec{u}))=-(\vec{v}+\vec{u})$$

$$\vec{w}+\quad\vec{0}\quad=-(1,7,3,7)$$

$$\boxed{\vec{w}=(-1,-7,-3,-7)}$$

b. $\mathbf{w}+3\mathbf{v}=-2\mathbf{u}$

$$\vec{w}=-2\vec{u}+(-3\vec{v})$$

$$\vec{w}=(-4,2,-6,-8)+(3,-24,0,-9)$$

$$\boxed{\vec{w}=(-1,-22,-6,-17)}$$

LINEAR COMBINATIONS OF VECTORS

An important type of problem in linear algebra involves writing one vector as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The vector $\vec{x}$, $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Example 5: If possible, write $\mathbf{v}$ as a linear combination of $\mathbf{u}$ and $\mathbf{w}$, where $\mathbf{u} = (1,2)$ and $\mathbf{w} = (1,-1)$.

a. $\mathbf{v} = (1,-1)$

$$\boxed{0\vec{u} + 1\vec{w} = \vec{v}}$$

$$c_1\vec{u} + c_2\vec{w} = \vec{v}$$

$$c_1(1,2) + c_2(1,-1) = (1,-1)$$

$$\left.\begin{aligned} c_1(1) + c_2(1) &= 1 \\ c_1(2) + c_2(-1) &= -1 \end{aligned}\right\}$$

$$3c_1 = 0$$

$$c_1 = 0,\ c_2 = 1$$

b. $\mathbf{v} = (0,3)$

$$\boxed{1\vec{u} - 1\vec{w} = \vec{v}}$$

$$c_1\vec{u} + c_2\vec{w} = \vec{v}$$

$$c_1(1,2) + c_2(1,-1) = (0,3)$$

$$1c_1 + 1c_2 = 0$$

$$2c_1 - 1c_2 = 3$$

$$3c_1 = 3$$

$$c_1 = 1,\ c_2 = -1$$

Example 6: If possible, write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

$\mathbf{v} = (-1,7,2)$, $\mathbf{u}_1 = (1,3,5)$, $\mathbf{u}_2 = (2,-1,3)$, and $\mathbf{u}_3 = (-3,2,-4)$.

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{v}$$

$$c_1(1,3,5) + c_2(2,-1,3) + c_3(-3,2,-4) = (-1,7,2)$$

$$\begin{aligned} 1c_1 + 2c_2 - 3c_3 &= -1 \\ 3c_1 - 1c_2 + 2c_3 &= 7 \\ 5c_1 + 3c_2 - 4c_3 &= 2 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no solution

It's not possible to write $\vec{v}$ as a linear combo of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$.

Section 4.2: VECTOR SPACES

When you are done with your homework you should be able to…

- π Define a vector space and recognize some important vector spaces
- π Show that a given set is not a vector space

VECTOR SPACE

A vector space consists of four entities: a set of vectors, a set of scalars, and two operations. When you refer to a vector space V, be sure that all four entities are clearly stated or understood. Unless stated otherwise, assume that the set of scalars is the set of real numbers.

IMPORTANT VECTOR SPACES CONTINUED

$C(-\infty,\infty)$ = the set of all continuous functions defined on the real number line.

$C[a,b]$ = the set of all continuous functions defined on a closed interval $[a,b]$.

P = the set of all polynomials.

P_n = the set of all polynomials of degree $\leq n$.

$M_{m,n}$ = the set of all $m \times n$ matrices.

$M_{n,n}$ = the set of all $n \times n$ square matrices.

Example 1: Describe the zero vector (the additive identity) of the vector space.

a. $C(-\infty,\infty)$

$f(x) = 0$

b. $M_{1,4}$

$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

Example 2: Describe the additive inverse of a vector in the vector space.

a. $C(-\infty,\infty)$ (the set of all real-valued continuous functions defined on the entire real line.

$f(x) + [-f(x)] = 0$

$\boxed{-f(x)}$

b. $M_{1,4}$

$M_{1,4} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$

$\boxed{\begin{bmatrix} -v_1 & -v_2 & -v_3 & -v_4 \end{bmatrix}}$

DEFINITION OF A VECTOR SPACE

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition

1. $\mathbf{u}+\mathbf{v}$ is in V. closure under addition

2. $\mathbf{u}+\mathbf{v} = \vec{v}+\vec{u}$ commutative property

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\vec{u}+\vec{v})+\vec{w}$ Associative property

4. V has a zero vector $\vec{0}$

   such that for every $\vec{v}$ in V, $\vec{v}+\vec{0}=\vec{v}$. additive identity

5. For every $\vec{v}$ in V, there is a vector in

   V denoted by $-\vec{v}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. additive inverse

Scalar Multiplication

6. $c\mathbf{u}$ is in V. closure under scalar mult.

7. $c(\mathbf{u}+\mathbf{v}) = c\vec{u}+c\vec{v}$ Distributive property

8. $(c+d)\mathbf{u} = c\vec{u}+d\vec{u}$ Distributive property

9. $c(d\mathbf{u}) = (cd)\vec{u}$ Associative property

10. $1(\mathbf{u}) = \vec{u}$ multiplicative identity

THEOREM 4.4: PROPERTIES OF SCALAR MULTIPLICATION

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} =$ $\vec{0}$________
2. $c\mathbf{0} =$ $\vec{0}$________
3. If $c\vec{v} = \vec{0}$____________, then $c=0$____________ or $\vec{v}=\vec{0}$____________.
4. $(-1)\mathbf{v} =$ ____$-\vec{v}$____

Example 3: Determine whether the set, together with the indicated operations, is a vector space. If it is not, then identify at least one of the ten vector space axioms that fails.

a. The set of all 2 x 2 matrices of the form

$$\begin{bmatrix} a & b \\ c & 1 \end{bmatrix}$$

No.

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix}$$

Fails closure under addition

b. The set $\left\{\left(x,\frac{1}{2}x\right): x\in\mathbb{R}\right\}$.

Let $\vec{x}_1=(x_1,\frac{1}{2}x_1)$,
$\vec{x}_2=(x_2,\frac{1}{2}x_2)$,
$\vec{x}_3=(x_3,\frac{1}{2}x_3)$, $x_1,x_2,x_3,c,d\in\mathbb{R}$

1) Closure (+):

$$\begin{aligned}\vec{x}_1+\vec{x}_2&=(x_1,\tfrac{1}{2}x_1)+(x_2,\tfrac{1}{2}x_2)\\&=(x_1+x_2,\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)\\&=(x_1+x_2,\tfrac{1}{2}(x_1+x_2))\ \checkmark\end{aligned}$$

2) Comm. under +:

$$\begin{aligned}\vec{x}_1+\vec{x}_2&=(x_1,\tfrac{1}{2}x_1)+(x_2,\tfrac{1}{2}x_2)\\&=(x_1+x_2,\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)\\&=(x_2+x_1,\tfrac{1}{2}x_2+\tfrac{1}{2}x_1)\\&=(x_2,\tfrac{1}{2}x_2)+(x_1,\tfrac{1}{2}x_1)\\&=\vec{x}_2+\vec{x}_1\ \checkmark\end{aligned}$$

3) Assoc. (+):

$$\begin{aligned}\vec{x}_1+(\vec{x}_2+\vec{x}_3)&=(x_1,\tfrac{1}{2}x_1)+\left[(x_2,\tfrac{1}{2}x_2)+(x_3,\tfrac{1}{2}x_3)\right]\\&=(x_1,\tfrac{1}{2}x_1)+(x_2+x_3,\tfrac{1}{2}x_2+\tfrac{1}{2}x_3)\\&=(x_1+(x_2+x_3),\tfrac{1}{2}x_1+(\tfrac{1}{2}x_2+\tfrac{1}{2}x_3))\\&=((x_1+x_2)+x_3,(\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)+\tfrac{1}{2}x_3)\\&=(x_1+x_2,\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)+(x_3,\tfrac{1}{2}x_3)\\&=\left[(x_1,\tfrac{1}{2}x_1)+(x_2,\tfrac{1}{2}x_2)\right]+(x_3,\tfrac{1}{2}x_3)\\&=(\vec{x}_1+\vec{x}_2)+\vec{x}_3\ \checkmark\end{aligned}$$

4) + Identity:

$$\begin{aligned}\vec{x}_1+\vec{0}&=(x_1,\tfrac{1}{2}x_1)+(0x_1,0(\tfrac{1}{2}x_1))\\&=(x_1,\tfrac{1}{2}x_1)+(0,0)\\&=(x_1+0,\tfrac{1}{2}x_1+0)\\&=(x_1,\tfrac{1}{2}x_1)\\&=\vec{x}_1\ \checkmark\end{aligned}$$

5) + Inverse: $\vec{x}_1 + (-\vec{x}_1) = (x_1, \frac{1}{2}x_1) + [-1(x_1, \frac{1}{2}x_1)]$

$= (x_1, \frac{1}{2}x_1) + (-x_1, -\frac{1}{2}x_1)$

$= (x_1 + (-x_1), \frac{1}{2}x_1 + (-\frac{1}{2}x_1))$

$= (0, 0)$

$= \vec{0}$ ✓

6) Closure (mult): $c\vec{x}_1 = c(x_1, \frac{1}{2}x_1)$

$= (cx_1, c(\frac{1}{2}x_1))$

$= (cx_1, \frac{1}{2}(cx_1))$ ✓

7) Dist (+): $c(\vec{x}_1 + \vec{x}_2] = c[(x_1, \frac{1}{2}x_1) + (x_2, \frac{1}{2}x_2)]$

$= c(x_1 + x_2, \frac{1}{2}x_1 + \frac{1}{2}x_2)$

$= (c(x_1 + x_2), c(\frac{1}{2}x_1 + \frac{1}{2}x_2))$

$= (cx_1 + cx_2, c(\frac{1}{2}x_1) + c(\frac{1}{2}x_2))$

$= (cx_1, c(\frac{1}{2}x_1)) + (cx_2, c(\frac{1}{2}x_2))$

$= c(x_1, \frac{1}{2}x_1) + c(x_2, \frac{1}{2}x_2)$

$= c\vec{x}_1 + c\vec{x}_2$ ✓

8) Dist. (+): $(c+d)\vec{x}_1 = (c+d)(x_1, \frac{1}{2}x_1)$

$= ((c+d)x_1, (c+d)\frac{1}{2}x_1)$

$= (cx_1 + dx_1, c(\frac{1}{2}x_1) + d(\frac{1}{2}x_1))$

$= (cx_1, c(\frac{1}{2}x_1)) + (dx_1, d(\frac{1}{2}x_1))$

$= c(x_1, \frac{1}{2}x_1) + d(x_1, \frac{1}{2}x_1)$

$= c\vec{x}_1 + d\vec{x}_1$ ✓

9) Assoc (mult): $c(d\vec{x}_1) = c\left(d(x_1, \frac{1}{2}x_1)\right)$

$= c\left(dx_1, d(\frac{1}{2}x_1)\right)$

$= \left(c(dx_1), c[d(\frac{1}{2}x_1)]\right)$

$= \left((cd)x_1, (cd)(\frac{1}{2}x_1)\right)$

$= (cd)(x_1, \frac{1}{2}x_1)$

$= (cd)\vec{x}_1$ ✓

yes; $\{(x, \frac{1}{2}x) : x \in \mathbb{R}\}$ is a vector space.

10) mult. identity: $1\vec{x}_1 = 1(x_1, \frac{1}{2}x_1)$

$= \left(1x_1, 1(\frac{1}{2}x_1)\right)$

$= (x_1, \frac{1}{2}x_1)$

$= \vec{x}_1$ ✓

c. The set of all 2 x 2 nonsingular matrices with the standard operations.

No. Nonsingular matrices have a nonzero determinant.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 7 & 6 \end{bmatrix}$

$\det(A) = -2 \neq 0$ and $\det(B) = 7 \neq 0$.

$\det(A+B) = \det\begin{bmatrix} 1 & 1 \\ 10 & 10 \end{bmatrix} = 0$.

not closed under addition.

Example 4: Rather than use the standard definitions of addition and scalar multiplication in R^3, suppose these two operations are defined as stated below. With these new definitions, is R^3 a vector space?

a.

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$c(x, y, z) = (cx, cy, 0)$$

No; no multiplicative identity since

$$1(1,2,3) = (1\cdot 1, 1\cdot 2, 0)$$
$$= (1,2,0)$$
$$\neq (1,2,3)$$

b.

$$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2+1,y_1+y_2+1,z_1+z_2+1)$$

$$c(x,y,z)=(cx+c-1,cy+c-1,cz+c-1)$$

Addition

1) closure: $(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2+1,y_1+y_2+1,z_1+z_2+1)$ which has components that are elements of R^3. ✓

2) comm: $(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2+1,y_1+y_2+1,z_1+z_2+1)$

$=(x_2+x_1+1,y_2+y_1+1,z_2+z_1+1)$

$=(x_2,y_2,z_2)+(x_1,y_1,z_1)$ ✓

3) Assoc:

$(x_1,y_1,z_1)+[(x_2,y_2,z_2)+(x_3,y_3,z_3)]=(x_1,y_1,z_1)+(x_2+x_3+1,y_2+y_3+1,z_2+z_3+1)$

$=(x_1+(x_2+x_3+1),y_1+(y_2+y_3+1),z_1+(z_2+z_3+1))$

$=((x_1+x_2+1)+x_3,(y_1+y_2+1)+y_3,(z_1+z_2+1)+z_3)$

$=(x_1+x_2+1,y_1+y_2+1,z_1+z_2+1)+(x_3,y_3,z_3)$

$=[(x_1,y_1,z_1)+(x_2,y_2,z_2)]+(x_3,y_3,z_3)$ ✓

4) Identity: $\vec{0}=(-1,-1,-1)$

$(x_1,y_1,z_1)+(-1,-1,-1)=(x_1+(-1)+1,y_1+(-1)+1,z_1+(-1)+1)$

$=(x_1,y_1,z_1)$ ✓

5) Inverse: $-(x,y,z)=(-x-2,-y-2,-z-2)$

$(x_1,y_1,z_1)+[-(x_1,y_1,z_1)]=(x_1,y_1,z_1)+(-x_1-2,-y_1-2,-z_1-2)$

$=(x_1+(-x_1-2)+1,y_1+(-y_1-2)+1,z_1+(-z_1-2)+1)$

$=(-1,-1,-1)$

$=\vec{0}$ ✓

Scalar Mult.

6) Closure: $c(x_1,y_1,z_1)=(cx_1+c-1,cy_1+c-1,cz_1+c-1)$ which has components which are elements of the real numbers ✓

7) dist: $c[(x_1,y_1,z_1)+(x_2,y_2,z_2)]$

$$=c(x_1+x_2+1,y_1+y_2+1,z_1+z_2+1)$$

$$=(c(x_1+x_2+1)+c-1,c(y_1+y_2+1)+c-1,c(z_1+z_2+1)+c-1)$$

$$=(cx_1+c-1+cx_2+c-1+1,cy_1+c-1+cy_2+c-1+1,cz_1+c-1,cz_2+c-1+1)$$

$$=(cx_1+c-1,cy_1+c-1,cz_1+c-1)+(cx_2+c-1,cy_2+c-1,cz_2+c-1)$$

$$=c(x_1,y_1,z_1)+c(x_2,y_2,z_2)$$ ✓

8) dist: $(c+d)(x_1,y_1,z_1)=((c+d)(x_1)+(c+d)-1,(c+d)y_1+(c+d)-1,(c+d)z_1+(c+d)-1)$

$$=(cx_1+c-1+dx_1+d-1+1,cy_1+c-1+dy_1+d-1+1,$$
$$cz_1+c-1+dz_1+d-1+1)$$

$$=c(x_1,y_1,z_1)+d(x_1,y_1,z_1)$$ ✓

9) Assoc: $c(d(x_1,y_1,z_1))=c(dx_1+d-1,dy_1+d-1,dz_1+d-1)$

$$=(c(dx_1+d-1)+c-1,c(dy_1+d-1)+c-1,c(dz_1+d-1)+c-1)$$

$$=((cd)x_1+cd-1,(cd)y_1+cd-1,(cd)z_1+cd-1)$$

$$=(cd)(x_1,y_1,z_1)$$ ✓

10) Identity: $1(x_1,y_1,z_1)=(1x_1+1-1,1y_1+1-1,1z_1+1-1)$

$$=(x_1,y_1,z_1)$$ ✓

Section 4.3: SUBSPACES OF VECTOR SPACES

When you are done with your homework you should be able to…

- π Determine whether a subset W of a vector space V is a subspace of V
- π Determine subspaces of R^n

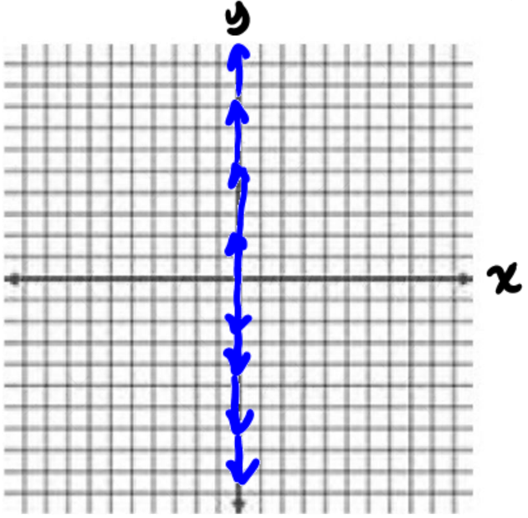
SUBSPACES

In many applications of linear algebra, vector spaces occur as a Subspace of larger spaces. A nonempty subset of a vector space is a subspace when it is a vector space with the same operations defined in the original vector space.

subset

$W \subseteq V$

Consider the following: $W = (0, y)$ and $V = R^2$.



$\vec{w}_1 = (0, 1)$

$\vec{w}_2 = (0, 5000)$

$\vec{w}_3 = (0, -3.5)$

examples of vectors in W.

DEFINITION OF A SUBSPACE OF A VECTOR SPACE

A nonempty subset W of a vector space V is called a Subspace of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

THEOREM 4.5: TEST FOR A SUBSPACE

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then __$\vec{u}+\vec{v}$__________ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then _$c\vec{u}$_________ is in W.

Example 1: Verify that W is a subspace of V.

a. $W=\{(x,y,2x-3y): x \text{ and } y\in\mathbb{R}\}$

$V=R^3$

W is a nonempty subset of V. Let $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ u_1,u_2,u_3,v_1,v_2,v_3, and $c\in\mathbb{R}$. $\vec{u},\vec{v}\in W$.

closure under addition: $\vec{u}+\vec{v}=(u_1,u_2,2u_1-3u_2)+(v_1,v_2,2v_1-3v_2$

$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,(2u_1-3u_2)+(2v_1-3v_2))$

$\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,(2u_1+2v_1)+(-3u_2+-3v_2))$

$u+\vec{v}=(u_1+v_1,u_2+v_2,2(u_1+v_1)-3(u_2+v_2))$ ✓

Closure under scalar mult:

$c\vec{u}=c(u_1,u_2,2u_1-3u_2)$

$c\vec{u}=(cu_1,cu_2,c(2u_1-3u_2))$

$c\vec{u}=(cu_1,cu_2,c2u_1-c3u_2)\rightarrow c\vec{u}=(cu_1,cu_2,2(cu_1)-3(cu_2))$. ✓

b. W is the set of all functions that are differentiable on $[-1,1]$. V is the set of all functions that are continuous on $[-1,1]$.

Since W is nonempty, and continuity implies differentiability, $W\subseteq V$. Let f and g be differentiable functions of x, $c\in\mathbb{R}$.

$\frac{d}{dx}f(x)+\frac{d}{dx}g(x)=\frac{d}{dx}(f(x)+g(x))$ ✓ ∴ Closure under +

$c\frac{d}{dx}f(x)=\frac{d}{dx}[cf(x)]$. ✓ closure under scalar mult.

∴ W is a subspace of V.

Example 2: Verify that W is not a subspace of the vector space by giving a specific example that violates the test for vector subspace.

Let. a. W is the set of all linear functions $ax+b$, $a \neq 0$ in $C(-\infty,\infty)$.

$y_1 = 2x+5$ and $y_2 = -2x+12$ are linear functions in W.

$y_1 + y_2 = (2x+5)+(-2x+12)$

$y_1 + y_2 = 0x+17$ which $\notin$ of W. Fails closure under addition. So W is not a subspace of V.

b. W is the set of all matrices in $M_{3,1}$, of the form $\begin{bmatrix} a & 0 & \sqrt{a} \end{bmatrix}^T$.

$$\begin{bmatrix} 9 \\ 0 \\ \sqrt{9} \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ \sqrt{4} \end{bmatrix} = \begin{bmatrix} \boxed{13} \\ 0 \\ \sqrt{\boxed{25}} \end{bmatrix}$$

$13 \neq 25$, so fails closure under addition.

So W is not a subspace.

THEOREM 4.6: THE INTERSECTION OF TWO SUBSPACES IS A SUBSPACE

If V and W are both subspaces of a vector space U, then the intersection of V and W, denoted by $V \cap W$, is also a subspace of U.

Example 3: Determine whether the subset $C(-\infty,\infty)$ is a subspace of $C(-\infty,\infty)$.

a. The set of all negative functions: $f(x) < 0$.

Let $c=-2$, $f(x)=-5+\sin x$. So $cf(x) = -2(-5+\sin x)$

$= 10-2\sin x > 0$.

So $f(x)<0$ is not a subspace.

b. The set of all odd functions: $f(-x) = -f(x)$.

The odd functions are a nonempty subset of $C(-\infty,\infty)$.

Let f and g be odd function. Let $c \in \mathbb{R}$

Closure under $+$: $(f+g)(-x) = f(-x)+g(-x)$

$$= -f(x)-g(x)$$

$$= -(f(x)+g(x))$$

$$= -(f+g)(x). \checkmark$$

Closure under scalar mult:

$$(cf)(-x) = c[f(-x)] = c[-f(x)] = -cf(x) \checkmark$$

∴ The set of all odd functions is a subspace of $C(-\infty,\infty)$.

Example 4: Determine whether the subset of $M_{n,n}$ is a subspace of $M_{n,n}$ with the standard operations of matrix addition and scalar multiplication.

a. The set of all n x n diagonal matrices.

Let $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \vdots & 0 \cdots & 0 & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & 0 & \cdots & \cdots & 0 \\ 0 & b_{22} & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots 0 & \cdots & b_{nn} \end{bmatrix}$, $a_{ij}, b_{ij}, c \in \mathbb{R}$.

Closure under +:

$A+B = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 \cdots & 0 & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{nn}+b_{nn} \end{bmatrix}$ ✓

Closure under scalar mult: $cA = c\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 \cdots & 0 & a_{nn} \end{bmatrix} = \begin{bmatrix} ca_{11} & 0 & \cdots & 0 \\ 0 & ca_{22} & 0 \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 \cdots & 0 & ca_{nn} \end{bmatrix}$ ✓

∴ The set of all nxn diag. mat. is a subspace of $M_{n,n}$.

b. The set of all n x n matrices whose trace is nonzero.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 \\ 6 & -4 \end{bmatrix}$

$A+B = \begin{bmatrix} 0 & 7 \\ 9 & 0 \end{bmatrix}$ which has a trace of zero. (closure under + fails)

$\boxed{\text{Not a subspace.}}$

Example 5: Determine whether the set W is a subspace of R^3 with the standard operations. Justify your answer.

$$W = \{(x_1, x_2, 4) : x_1 \text{ and } x_2 \in \mathbb{R}\}$$

$\vec{x}_1 = (1,1,4)$, $\vec{x}_2 = (2,2,4)$

$\vec{x}_1 + \vec{x}_2 = (3,3,8)$. Not closed under +. $\boxed{\text{Not a subspace.}}$

Section 4.4: SPANNING SETS AND LINEAR INDEPENDENCE

When you are done with your homework you should be able to…

- π Write a linear combination of a set of vectors in a vector space V
- π Determine whether a set S of vectors in a vector space V is a spanning set of V
- π Determine whether a set of vectors in a vector space V is linearly independent

DEFINITION OF LINEAR COMBINATION OF VECTORS IN A VECTOR SPACE

A vector $\mathbf{v}$ in a vector space V is called a __linear_______________ combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$$

where $c_1, c_2, \ldots c_k$ are scalars.

Example 1: If possible, write each vector as a linear combination of the vectors in S.

$$S = \{\overset{\vec{u}_1}{(1,2,-2)}, \overset{\vec{u}_2}{(2,-1,1)}\}$$

a. $\mathbf{z} = \overset{\vec{v}}{(-4,-3,3)}$

$$c_1(1,2,-2) + c_2(2,-1,1) = (-4,-3,3)$$

$$\begin{aligned} 1c_1 + 2c_2 &= -4 \\ 2c_1 - 1c_2 &= -3 \\ -2c_1 + 1c_2 &= 3 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 2 & -1 & -3 \\ -2 & 1 & 3 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} c_1 &= -2 \\ c_2 &= -1 \end{aligned}$$

$$\boxed{-2(1,2,-2) - (2,-1,1) = (-4,-3,3)}$$

$S=\{(1,2,-2),(2,-1,1)\}$

b. $\mathbf{u}=(1,1,-1)$

$c_1(1,2,-2)+c_2(2,-1,1)=(1,1,-1)$

$$\begin{aligned}1c_1+2c_2&=1\\2c_1-1c_2&=1\\-2c_1+1c_2&=-1\end{aligned}\rightarrow\left[\begin{array}{cc|c}1&2&1\\2&-1&1\\-2&1&-1\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&3/5\\0&1&1/5\\0&0&0\end{array}\right]\quad c_1=\frac{3}{5}\quad c_2=\frac{1}{5}$$

$$\boxed{\frac{3}{5}(1,2,-2)+\frac{1}{5}(2,-1,1)=(1,1,-1)}$$

Example 2: For the matrices

$$A=\begin{bmatrix}2&-3\\4&1\end{bmatrix}\text{ and }B=\begin{bmatrix}0&5\\1&-2\end{bmatrix}$$

in $M_{2,2}$, determine whether the given matrix is a linear combination of A and B.

$$\begin{bmatrix}6&-19\\10&7\end{bmatrix}$$

$$c_1\begin{bmatrix}2&-3\\4&1\end{bmatrix}+c_2\begin{bmatrix}0&5\\1&-2\end{bmatrix}=\begin{bmatrix}6&-19\\10&7\end{bmatrix}$$

$$\begin{aligned}2c_1+0c_2&=6\\-3c_1+5c_2&=-19\\4c_1+1c_2&=10\\1c_1-2c_2&=7\end{aligned}\rightarrow\left[\begin{array}{cc|c}2&0&6\\-3&5&-19\\4&1&10\\1&-2&7\end{array}\right]\xrightarrow{rref}\text{error CRAP!}$$

$$\searrow\left[\begin{array}{ccc|c}2&0&0&6\\-3&5&0&-19\\4&1&0&10\\1&-2&0&7\end{array}\right]$$

or

$$\begin{bmatrix}2&0&6\\-3&5&-19\\5&-1&17\end{bmatrix}\quad\text{Sum}$$

$c_1=3,\ c_2=-2$

$$3\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix}-2\begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix}=\begin{bmatrix}6 & -19\\ 10 & 7\end{bmatrix}$$

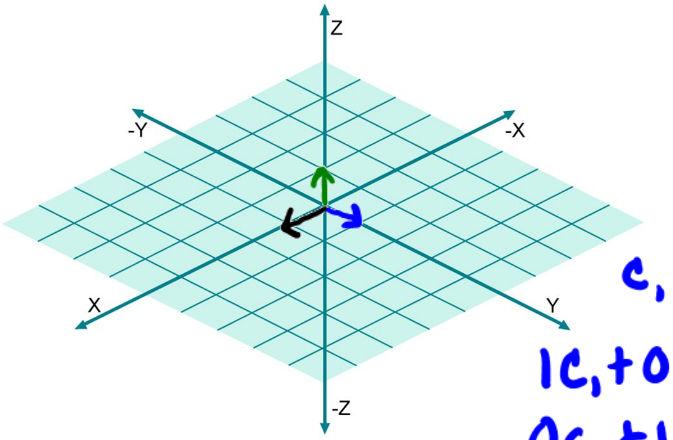
DEFINITION OF A SPANNING SET OF A VECTOR SPACE

Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a

__spanning______ set of V when _every____ vector in V can be written as

a __linear__________ __combination________ of vectors in S.

* In such cases, it is said that S spans V.



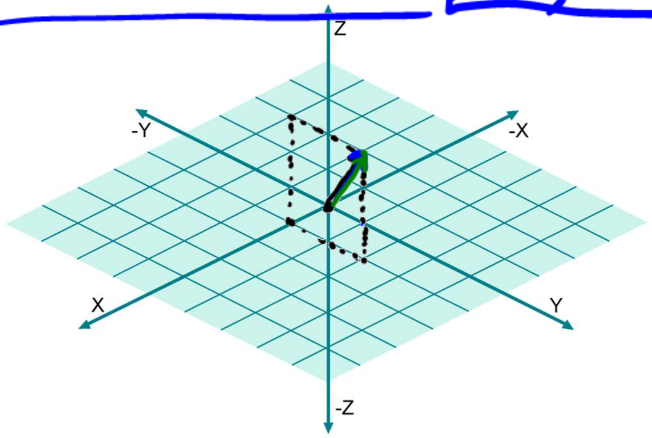
$S=\{(1,0,0),(0,1,0),(0,0,1)\}$

If $\vec{u}=(u_1,u_2,u_3)$ is any vector in R^3.

$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(u_1,u_2,u_3)$

$$\begin{aligned}1c_1+0c_2+0c_3&=u_1\\ 0c_1+1c_2+0c_3&=u_2\\ 0c_1+0c_2+1c_3&=u_3\end{aligned}\rightarrow c_1=u_1,\ c_2=u_2,\ c_3=u_3$$

$$\boxed{\begin{array}{l}u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)=\vec{u}\\ \text{So, } S \text{ spans } R^3.\end{array}}$$



$S=\{(1,2,3),(0,1,2),(-1,0,1)\}$

coplanar

$$\begin{vmatrix}1 & 0 & -1\\ 2 & 1 & 0\\ 3 & 2 & 1\end{vmatrix}=0\Rightarrow \text{coplanar}$$

S spans the plane in R^3 where these vectors lie. S does not span R^3.

DEFINITION OF THE SPAN OF A SET

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the span__________ of S is the set of all linear______________ combinations of the vectors in S.

$$\text{span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

The span of S is denoted by $\text{span}(S)$ or $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$

When $\text{span}(S) = V$______________, it is said that V is spanned______________ by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$______________, or that S spans V.

THEOREM 4.7: Span(*S*) IS A SUBSPACE OF *V*

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of a vectors in a vector space V, then $\text{span}(S)$ is a subspace of V. Moreover, $\text{span}(S)$ is the Smallest______________ subspace of V that contains S, in the sense that every other subspace of V that contains S must contain $\text{span}(S)$.

Proof:

In text

Example 3: Determine whether the set S spans R^2. If the set does not span R^2, then give a geometric description of the subspace that it does span.

a. $S=\{(1,-1),(2,1)\}$

Let $\vec{u}=(u_1,u_2)$ be any vector in R^2.

$c_1(1,-1)+c_2(2,1)=(u_1,u_2)$

$1c_1+2c_2=u_1$

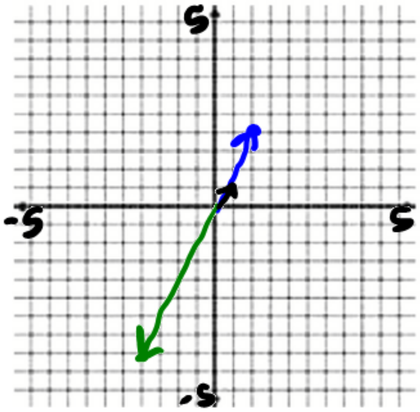
$-1c_1+1c_2=u_2$

$\begin{vmatrix}1 & 2\\ -1 & 1\end{vmatrix}=3\neq 0$ so there's a unique solution.

$\boxed{\therefore S \text{ spans } R^2.}$

b. $S=\left\{(1,2),(-2,-4),\left(\frac{1}{2},1\right)\right\}$

$\boxed{S \text{ spans the line } y=2x\text{, but not } R^2.}$



c. $S=\{(-1,2),(2,-1),(1,1)\}$

Let $\vec{u}=(u_1,u_2)$.

$c_1(-1,2)+c_2(2,-1)+c_3(1,1)=(u_1,u_2)$

$-c_1+2c_2+c_3=u_1$ R_1

$2c_1-c_2+c_3=u_2$ R_2

$2R_1+R_2$: $-2c_1+4c_2+2c_3=2u_1$

$\underline{2c_1-c_2+c_3=u_2}$

$3c_2+3c_3=2u_1+u_2$

Let $c_3=0$: $c_2=\frac{1}{3}(2u_1+u_2)$

From the original: $c_1=2c_2+c_3-u_1$

$c_1=2\left(\frac{1}{3}(2u_1+u_2)\right)+0-u_1\rightarrow c_1=\frac{2}{3}(2u_1+u_2)-u_1$

$c_1 = \frac{1}{3}u_1 + \frac{2}{3}u_2$, $c_2 = \frac{2}{3}u_1 + \frac{1}{3}u_2$

Let $\vec{u} = (\underset{\substack{\uparrow \\ u_1}}{2}, \underset{\substack{\uparrow \\ u_2}}{3})$

$c_1 = \frac{1}{3}(2) + \frac{2}{3}(3)$ $c_2 = \frac{2}{3}(2) + \frac{1}{3}(3)$

$c_1 = \frac{8}{3}$ $c_2 = \frac{7}{3}$

$$\frac{8}{3}(-1,2) + \frac{7}{3}(2,-1) = (2,3)$$

DEFINITION OF LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

A set of vector $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called linearly

independent when the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0$$

If there are also nontrivial solutions, then S is called linearly

dependent.

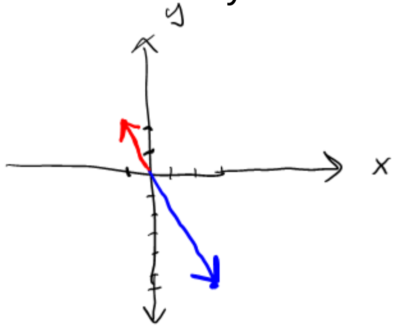
TESTING FOR LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent of linearly dependent, use the following steps.

1. From the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$, write a system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system has nontrivial solutions, then S is linearly dependent.

Example 4: Determine whether the set S is linearly independent or linearly dependent.

a. $S=\{(3,-6),(-1,2)\}$



$c_1(3,-6)+c_2(-1,2) = (0,0)$

$3c_1 - c_2 = 0 \qquad\qquad 6c_1-2c_2=0$

$-6c_1+2c_2=0 \longrightarrow \underline{-6c_1+2c_2=0}$

$0=0$

infinitely many solutions

$\boxed{\text{S is linearly dependent.}}$

b. $S=\{(6,2,1),(-1,3,2)\}$

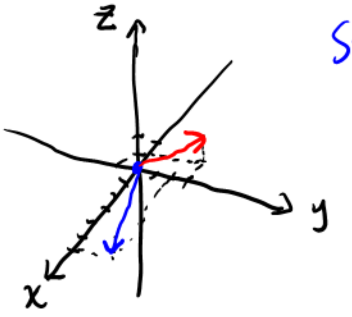
$6c_1 - c_2 = 0$

$2c_1+3c_2=0 \longrightarrow 2(-2c_2)+3c_2=0 \rightarrow -c_2=0 \rightarrow c_2=0$

$c_1+2c_2=0 \longrightarrow c_1=-2c_2 \rightarrow c_1=-2(0) \rightarrow c_1=0$

only the trivial solution

$\boxed{\text{S is linearly independent}}$



c. $S=\{(0,0,0,1),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$

$c_4=0$

$c_3+c_4=0 \rightarrow c_3=0$

$c_2+c_3+c_4=0 \rightarrow c_2=0$

$c_1+c_2+c_3+c_4=0 \rightarrow c_1=0$

only the trivial solution

$\boxed{\text{S is linearly independent}}$

Example 5: Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$S = \{x^2, x^2 + 1\}$

$c_1x^2 + c_2(x^2+1) = 0 + 0x + 0x^2$

$c_1x^2 + c_2x^2 + c_2 = 0 + 0x + 0x^2$

$(c_1+c_2)x^2 + c_2 = 0 + 0x + 0x^2$

$c_1 + c_2 = 0 \rightarrow c_1 = 0$

$c_2 = 0$

only the trivial solution

$\boxed{S \text{ is linearly independent}}$

Example 6: Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix}$

$c_1\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + c_2\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} + c_3\begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$2c_1 - 4c_2 - 8c_3 = 0 \rightarrow 2(-2c_3) - 4(-3c_3) - 8c_3 = 0 \rightarrow 0c_3 = 0 \rightarrow 0 = 0$

$-1c_2 - 3c_3 = 0 \rightarrow c_2 = -3c_3$

$-3c_1 - 6c_3 = 0 \rightarrow c_1 = -2c_3$

$1c_1 + 5c_2 + 17c_3 = 0$

infinitely many solutions

$\boxed{\text{This set is linearly dependent}}$

$\begin{bmatrix} 2 & -4 & -8 & 0 \\ 0 & -1 & -3 & 0 \\ -3 & 0 & -6 & 0 \\ 1 & 5 & 17 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

THEOREM 4.8: A PROPERTY OF LINEARLY DEPENDENT SETS

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

Proof:

In Text

THEOREM 4.8: COROLLARY

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a __scalar__________________ of the other.

Example 7: Show that the set is linearly dependent by finding a nontrivial linear combination of vectors in the set whose sum is the zero vector. Then express one of the vectors in the set as a linear combination of the other vectors in the set.

$S = \{(2,4),(-1,-2),(0,6)\}$

$c_1(2,4) + c_2(-1,-2) + c_3(0,6) = (0,0)$

$$\begin{aligned} 2c_1 - c_2 \quad\quad &= 0 \\ 4c_1 - 2c_2 + 6c_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 4 & -2 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{matrix} c_1 - \frac{1}{2}c_2 = 0 \rightarrow c_1 = \frac{1}{2}c_2 \\ c_3 = 0 \end{matrix}$$

$\frac{1}{2}c_2(2,4) + c_2(-1,-2) + 0(0,6) = (0,0)$

Let $c_2 = 2$

$1(2,4) + 2(-1,-2) + 0(0,6) = (0,0)$

S is Linearly dependent

Section 4.5: BASIS AND DIMENSION

When you are done with your homework you should be able to…

- π Recognize bases in the vector spaces R^n, P_n, and $M_{m,n}$
- π Find the dimension of a vector space

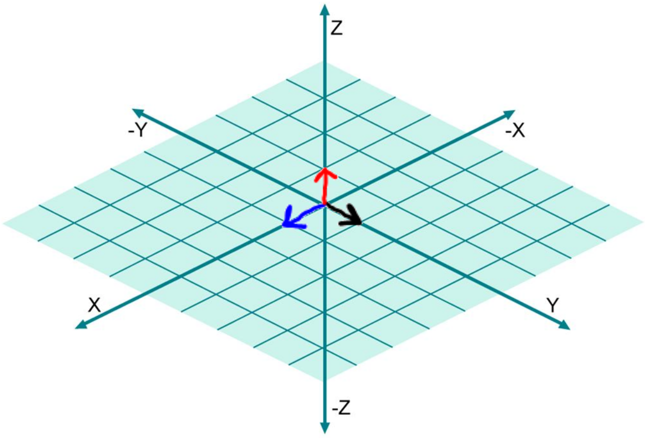
DEFINITION OF BASIS

A set of vectors $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ in a vector space V is called a basis for V when the following conditions are true.

1. S span V.
2. S is linearly independent.

The Standard Basis for R^3

$$S=\{(1,0,0),(0,1,0),(0,0,1)\}$$



1) Does S span R^3?

$\vec{u}=(u_1,u_2,u_3)$

$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(u_1,u_2,u_3)$

$1c_1+0c_2+0c_3=u_1 \rightarrow c_1=u_1$

$0c_1+1c_2+0c_3=u_2 \rightarrow c_2=u_2$

$0c_1+0c_2+1c_3=u_3 \rightarrow c_3=u_3$

$\vec{u}=u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1)$

So S spans R^3 ✓

2) Is S lin. ind.?

$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$

$c_1=0,\ c_2=0,\ c_3=0$

So S is linearly independent ✓

S is a basis for R^3 since S spans R^3 and S is linearly independent.

Example 1: Write the standard basis for the vector space.

a. R^5 $S = \{(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)\}$

b. $M_{5,2}$

$$\boxed{S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}$$

$$\begin{bmatrix} 10 & 9 \\ 8 & 7 \\ 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots + 1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

c. P_2

$$P_0(x) = 1x^0 + 0x^1 + 0x^2 = 1$$

$$P_1(x) = 0x^0 + 1x^1 + 0x^2 = x$$

$$P_2(x) = 0x^0 + 0x^1 + 1x^2 = x^2$$

$$\boxed{S = \{1, x, x^2\}}$$

Example 2: Determine whether S is a basis for the indicated vector space.

a. $S = \{(2,1,0),(0,-1,1)\}$ for R^3

1) Does S span R^3? $\vec{u} = (u_1, u_2, u_3)$

$c_1(2,1,0) + c_2(0,-1,1) = (u_1, u_2, u_3)$

$2c_1 = u_1 \rightarrow c_1 = \frac{1}{2}u_1$

$c_1 - c_2 = u_2 \rightarrow c_1 = u_2 + c_2 = u_2 + u_3$

$c_2 = u_3$

Let $\vec{u} = (1,2,3)$ $u_1 = 1,\ u_2 = 2,\ u_3 = 3$

$\frac{1}{2}u_1(2,1,0) + u_3(0,-1,1) = (u_1, u_2, u_3)$

$\frac{1}{2} \cdot 1(2,1,0) + 3(0,-1,1) \overset{?}{=} (1,2,3)$

$(1,\frac{1}{2},0) + (0,-3,3) \overset{?}{=} (1,2,3)$

$(1,-\frac{5}{2},3) \neq (1,2,3)$

b. $S=\left\{4t-t^2, 5+t^3, 3t+5, 2t^3-3t^2\right\}$ for P_3

1) Does S span P_3?

$$c_1(4t-t^2)+c_2(5+t^3)+c_3(3t+5)+c_4(2t^3-3t^2)=a_0t^0+a_1t^1+a_2t^2+a_3t^3$$

$$4c_1t-c_1t^2+5c_2+c_2t^3+3c_3t+5c_3+2c_4t^3-3c_4t^2=a_0t^0+a_1t^1+a_2t^2+a_3t^3$$

$$(5c_2+5c_3)t^0+(4c_1+3c_3)t^1+(-c_1-3c_4)t^2+(c_2+2c_4)t^3=a_0t^0+a_1t^1+a_2t^2+a_3t^3$$

$$\begin{array}{rrrrl} & 5c_2 & +5c_3 & & =a_0 \\ 4c_1 & & +3c_3 & & =a_1 \\ -c_1 & & & -3c_4 & =a_2 \\ & c_2 & & +2c_4 & =a_3 \end{array} \qquad \begin{vmatrix} 0 & 5 & 5 & 0 \\ 4 & 0 & 3 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{vmatrix} = 30 \neq 0 \Rightarrow \text{there's a unique solution.}$$

$\therefore$ S spans P_3. ✓

2) Is S linearly independent?

$$\begin{array}{rrrrl} & 5c_2 & +5c_3 & & =0 \\ 4c_1 & & +3c_3 & & =0 \\ -c_1 & & & -3c_4 & =0 \\ & c_2 & & +2c_4 & =0 \end{array} \rightarrow \left[\begin{array}{rrrr|r} 0 & 5 & 5 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

So $c_1=c_2=c_3=c_4=0$ ✓

$\boxed{\therefore S \text{ is a basis for } P_3.}$

THEOREM 4.9: UNIQUENESS OF BASIS REPRESENTATION

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof: 1) Assume S is a basis for V. So S spans V and S is linearly independent. ∃ a $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$. Let's suppose that we could also represent $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$.

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

$$\underline{-(\vec{u})} = \underline{-(b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n)}$$

$$\vec{0} = (c_1 - b_1)\vec{v}_1 + (c_2 - b_2)\vec{v}_2 + \cdots + (c_n - b_n)\vec{v}_n$$

$c_1 - b_1 = 0,\ c_2 - b_2 = 0,\ \ldots,\ c_n - b_n = 0$ [since S is linearly independent]

$c_1 = b_1,\ c_2 = b_2,\ \ldots,\ c_n = b_n$.

∴ $\vec{u}$ has only one representation for the basis S. ✓

Part 2 is in the text.

THEOREM 4.10: BASES AND LINEAR DEPENDENCE

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than __n____ vectors in V is linearly __dependent______________________.

THEOREM 4.11: NUMBER OF VECTORS IN A BASIS

If a vector space V has one basis with _n_ _vectors_, then every basis for V has _n_ vectors.

Proof:

Let $S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be the basis for V, and let $S_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be any other basis for V. S_1 is a basis and we know that S_2 is linearly independent, $m \le n$ [Thm 4.10]. S_1 is linearly independent and since S_2 is a basis, $n \le m$ [Thm. 4.10]. ∴ $n = m$. ✓

DEFINITION OF DIMENSION OF A VECTOR SPACE

If a vector space V has a _basis_ consisting of _n_ vectors, then the number _n_ is called the _dimension_ of V, denoted by _$\dim(V) = n$_. When V consists of the _zero_ vector alone, the dimension of V is defined as _zero_.

Example 3: Determine the dimension of the vector space.

a. R^5

$\boxed{\dim(R^5) = 5}$

b. $M_{5,2}$

$\boxed{\dim(M_{5,2}) = 10}$

c. P_2

$\boxed{\dim(P_2) = 3}$

THEOREM 4.12: BASIS TESTS IN AN *n*-DIMENSIONAL SPACE

Let V be a vector space of dimension n.

1. If $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n)$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ spans V, then S is a basis for V.

Proof:

In text

Example 4: Determine whether S is a basis for the indicated vector space.

$S = \{(1,2),(1,-1)\}$ for R^2.

$\dim(R^2) = 2$ since the standard basis is $\{(1,0),(0,1)\}$.

$c_1(1,2) + c_2(1,-1) = (0,0)$

$$\begin{aligned} c_1 + c_2 &= 0 \\ \underline{2c_1 - c_2} &= \underline{0} \\ 3c_1 &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

$\boxed{\text{Since } S \text{ has 2 linearly independent vectors, and } \dim(R^2)=2, \; S \text{ is a basis for } R^2 \text{ [by Thm 4.12].}}$

Example 5: Find a basis for the vector space of all 3 x 3 symmetric matrices. What is the dimension of this vector space?

3x3 symmetric matrix: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$a_{ij} = a_{ji}$

$a_{21} = a_{12}$

$a_{31} = a_{13}$

$a_{32} = a_{23}$

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

$\dim(\text{all } 3\times3 \text{ sym. mat.}) = 6$.

Section 4.6: RANK OF A MATRIX AND SYSTEMS OF LINEAR EQUATIONS

When you are done with your homework you should be able to…

- Find a basis for the row space, a basis for the column space, and the rank of a matrix
- Find the nullspace of a matrix
- Find the solution of a consistent system $A\mathbf{x}=\mathbf{b}$ in the form $\mathbf{x}_p+\mathbf{x}_h$

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

row vectors: $(a_{11}, a_{12}, a_{13}, \ldots, a_{1n})$

$(a_{21}, a_{22}, a_{23}, \ldots, a_{2n})$

$\vdots$

$(a_{m1}, a_{m2}, \ldots, a_{mn})$

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

column vectors: $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

or $(a_{11}, a_{21}, \ldots, a_{m1})^T, (a_{12}, a_{22}, \ldots, a_{m2})^T, \ldots,$
$(a_{1n}, a_{2n}, \ldots, a_{mn})^T$

Example 1: Consider the following matrix.

$$A=\begin{bmatrix} 1 & 3 & -1 & 5 \\ 7 & 1 & 13 & 6 \end{bmatrix}$$

a. The row vectors of A are:

$(1,3,-1,5), (7,1,13,6)$

b. The column vectors of A are:

$\begin{bmatrix} 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 13 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

DEFINITION OF ROW SPACE AND COLUMN SPACE OF A MATRIX

Let A be an $m \times n$ matrix.

1. The row space of A is the subspace of R^n spanned by the row vectors of A.
2. The column space of A is the subspace of R^n spanned by the column vectors of A.

Recall that two matrices are row-equivalent when one can be obtained from the other by elementary row *operations.*

THEOREM 4.13: ROW-EQUIVALENT MATRICES HAVE THE SAME ROW SPACE

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

Proof:

In text

THEOREM 4.14: BASIS FOR THE ROW SPACE OF A MATRIX

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a __basis__________ for the row space of A.

Example 2: Find a basis for the row space of the following matrix:

$$A=\begin{bmatrix}2&-3&1\\5&10&6\\8&-7&5\end{bmatrix}\overset{\text{OR}}{\xrightarrow{\text{rref}}}\begin{bmatrix}1&0&4/5\\0&1&1/5\\0&0&0\end{bmatrix}\begin{array}{l}\rightarrow\text{nonzero row}\\ \rightarrow\text{nonzero row}\\ \rightarrow\text{zero row}\end{array}$$

Basis for the row space of A:

$\{(2,-3,1),(5,10,6)\}$ OR $\{(1,0,4/5),(0,1,1/5)\}$

Example 3: Find a basis for the column space of the following matrix:

$$A = \begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix}$$

2 ways:

1) Find the basis for the row space of A^T.

$$A^T = \begin{bmatrix} 4 & 6 & 2 \\ 20 & -5 & -11 \\ 31 & -6 & -16 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for the column space of A is:

$$\{(4,6,2)^T, (20,-5,-11)^T\} \text{ OR } \{(1,0,-2/5)^T, (0,1,3/5)^T\}$$

2) Use the rref (A) to see which columns have leading 1's. Use these columns in the non-reduced matrix (original A) as the basis.

$$A = \begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\{(4,6,2)^T, (20,-5,-11)^T\}}$$

must use orig. columns that correspond to

leading 1's

THEOREM 4.15: ROW AND COLUMN SPACES HAVE EQUAL DIMENSIONS

If A is an $m \times n$ matrix, then the row space and the column space of A have the same __dimensions__________________.

DEFINITION OF THE RANK OF A MATRIX

The ___dimension__________ of the __row_____ (or __column__________) space of a matrix A is called the __rank_____ of A and is denoted by __rank(A)______. R(A)

Example 4: Find the rank of the matrix from

a. example 2, and

rank(A) = 2

b. example 3

rank (A) = 2

THEOREM 4.16: SOLUTIONS OF A HOMOGENEOUS SYSTEM

If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\vec{x} = \vec{0}$ is a subspace of R^n called the nullspace of A and is denoted $N(A)$. So,

$$N(A) = \{\vec{x} \in R^n : A\vec{x} = \vec{0}\}$$

The dimension of the nullspace of A is called the nullity of A.

Proof:

In text

Example 5: Find the nullspace of the following matrix A, and determine the nullity of A.

$$A=\begin{bmatrix}1 & 4 & 2 & 1\\ 0 & 1 & 1 & -1\\ -2 & -8 & -4 & -2\end{bmatrix} \rightarrow \left[\begin{array}{cccc|c}1 & 4 & 2 & 1 & 0\\ 0 & 1 & 1 & -1 & 0\\ -2 & -8 & -4 & -2 & 0\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c}1 & 0 & -2 & 5 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$x_1 \quad -2x_3+5x_4=0$$
$$x_2+x_3-x_4=0$$

let $x_3=s$, $x_4=t$

$x_1=2s-5t$, $x_2=-s+t$, $x_3=S$, $x_4=t$, $s,t\in R$

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2s-5t\\-s+t\\s\\t\end{bmatrix}=s\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-5\\1\\0\\1\end{bmatrix}$$

A basis for the nullspace of A is

$$\left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-5\\1\\0\\1\end{bmatrix}\right\}$$

and $N(A)=2$

THEOREM 4.17: DIMENSION OF THE SOLUTION SPACE

If A is an $m \times n$ matrix of rank __r__, then the __dimension__ of the solution space of __$A\vec{x} = \vec{0}$__ is __$n - r$__. That is,

$$n = \text{rank}(A) + \text{nullity}(A)$$

Example 6: consider the following homogeneous system of linear equations:

$$\begin{aligned} x - y &= 0 \\ -x + y &= 0 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a. Find a basis for the solution space.

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Let $x_2 = t \rightarrow x_1 = t, x_2 = t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A basis for the solution space of $A\vec{x} = 0$ is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

FYI: Since the equations were equal to 0, this is also a basis for the nullspace of A.

b. Find the dimension of the solution space.

$n = 2$ (2 columns in A)

rank(A) = 1

$\boxed{\text{dim of solution space is } n - r = 2 - 1 = 1}$

THEOREM 4.18: SOLUTIONS OF A NONHOMOGENEOUS LINEAR SYSTEM

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form _$\vec{x} = \vec{x}_p + \vec{x}_h$______ where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system_$A\vec{x} = \vec{0}$___.

Proof: Let $\vec{x}$ be any solution of $A\vec{x} = \vec{b}$. Then $(\vec{x} - \vec{x}_p)$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ since

$$A(\vec{x} - \vec{x}_p) = A\vec{x} - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0}.$$

Let $x_h = \vec{x} - \vec{x}_p \rightarrow \vec{x} = \vec{x}_p + \vec{x}_h$. ✓

THEOREM 4.19: SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

The system __$A\vec{x} = b$_____ is consistent if and only if _$\vec{b}$__ is in the column space of _A____.

Proof: For the system $A\vec{x} = \vec{b}$,

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

So $A\vec{x} = \vec{b}$ iff $\vec{b} = (b_1\ b_2 \cdots b_n)^T$ is a linear combination of the columns of A. That is, the system is consistent if and only if $\vec{b} \in$ subspace of R^m spanned by the columns of A. //

Example 7: consider the following nonhomogeneous system of linear equations:

$$\begin{aligned} 2x-4y+5z&=8\\ -7x+14y+4z&=-28\\ 3x-6y+z&=12 \end{aligned}$$

a. Determine whether $A\mathbf{x}=\mathbf{b}$ is consistent.

$$\left[\begin{array}{ccc|c} 2&-4&5&8\\ -7&14&4&-28\\ 3&-6&1&12 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1&-2&0&4\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix}$$

Column space of A:

$$\left\{ \begin{bmatrix}2\\-7\\3\end{bmatrix}, \begin{bmatrix}5\\4\\1\end{bmatrix} \right\}$$

$$\begin{aligned} x-2y&=4\\ z&=0 \end{aligned}$$

$$c_1(2,-7,3)+c_2(5,4,1)=(8,-28,12)$$

$$\begin{aligned} 2c_1+5c_2&=8\\ -7c_1+4c_2&=-28\\ 3c_1+c_2&=12 \end{aligned} \rightarrow \begin{bmatrix} 2&5&8&0\\ -7&4&-28&0\\ 3&1&12&0 \end{bmatrix} \rightarrow \begin{bmatrix} 1&0&4&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}$$

$c_1=4$, $c_2=0$

b. If the system is consistent, write the solution in the form $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x}=\mathbf{b}$ and $\mathbf{x}_h$ is a solution of $A\mathbf{x}=\mathbf{0}$.

since $4(2,-7,3)^T+0(5,4,1)^T=\vec{b}$, $b\in$ column space, ∴ the system is consistent.

Let $y=t$: $x=2t+4$, $y=t$, $z=0$

$$\vec{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2t+4\\t\\0\end{bmatrix}=t\begin{bmatrix}2\\1\\0\end{bmatrix}+\begin{bmatrix}4\\0\\0\end{bmatrix}$$

$$\boxed{\vec{x}=\underbrace{\begin{bmatrix}4\\0\\0\end{bmatrix}}_{\vec{x}_p}+\underbrace{t\begin{bmatrix}2\\1\\0\end{bmatrix}}_{\vec{x}_h}}$$

Section 4.7: COORDINATES AND CHANGE OF BASIS

When you are done with your homework you should be able to…

- Find a coordinate matrix relative to a basis in R^n
- Find the transition matrix from the basis B to the basis B' in R^n
- Represent coordinates in general *n*-dimensional spaces

COORDINATE REPRESENTATION RELATIVE TO A BASIS

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V, and let $\mathbf{x}$ be a vector in V such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\vec{x}$ relative to the basis B. The coordinate matrix (or coordinate vector) of $\vec{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\vec{x}$.

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note: In R^n, column notation is used for the coordinate matrix. For the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$, the x_i's are the coordinates of $\vec{x}$ (relative to the standard basis S for R^n. So you have

$$[\vec{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example 1: Find the coordinate matrix of $\mathbf{x}$ in R^n relative to the standard basis.

$\mathbf{x}=(1,-3,0)$

$$S=\{(1,0,0),(0,1,0),(0,0,1)\}$$

$$c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(1,-3,0)$$

$$c_1=1$$
$$c_2=-3$$
$$c_3=0$$

$$[\vec{x}]_S=\begin{bmatrix}1\\-3\\0\end{bmatrix}$$

Example 2: Given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis B for R^n, find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

$B=\{(4,0,7,3),(0,5,-1,-1),(-3,4,2,1),(0,1,5,0)\}$

$$[\mathbf{x}]_B=\begin{bmatrix}-2\\3\\4\\1\end{bmatrix}$$

$c_1=-2$
$c_2=3$
$c_3=4$
$c_4=1$

$$\vec{x}=-2(4,0,7,3)+3(0,5,-1,-1)+4(-3,4,2,1)+1(0,1,5,0)$$
$$\vec{x}=(-8,0,-14,-6)+(0,15,-3,-3)+(-12,16,8,4)+(0,1,5,0)$$
$$\vec{x}=(-20,32,-4,-5)$$

$x_1=-20,\ x_2=32,\ x_3=-4,\ x_4=-5$

$$\boxed{[\vec{x}]_S=\begin{bmatrix}-20\\32\\-4\\-5\end{bmatrix}}$$

Example 3: Find coordinate matrix of $\mathbf{x}$ in R^n relative to the basis B'.

$B' = \{(-6,7),(4,-3)\}$, $\mathbf{x} = (-26,32) \leftarrow [\vec{x}]_B$

$$\vec{x} = c_1(u_1,u_2) + c_2(u_1,u_2)$$

$$(-26,32) = c_1(-6,7) + c_2(4,-3)$$

$$\begin{aligned} -6c_1 + 4c_2 &= -26 \\ 7c_1 - 3c_2 &= 32 \end{aligned} \rightarrow \left[\begin{array}{cc|c} -6 & 4 & -26 \\ 7 & -3 & 32 \end{array}\right] \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{aligned} c_1 &= 5 \\ c_2 &= 1 \end{aligned}$$

$$\boxed{[\vec{x}]_{B'} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

The last two examples used the procedure called a ___change___ of ___basis___.

You were given the coordinates of a vector relative to a ___basis___ ___B___

and were asked to find the coordinates ___relative___ to another basis

___B'___.

$$\underbrace{\begin{bmatrix} -6 & 4 \\ 7 & -3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[\vec{x}]_{B'}} = \underbrace{\begin{bmatrix} -26 \\ 32 \end{bmatrix}}_{[\vec{x}]_B}$$

The matrix ___P___ is called the ___transition___ ___matrix___ from ___B'___ to ___B___, where ___$[\vec{x}]_{B'}$___ is the coordinate matrix of ___$\vec{x}$___ relative to ___B'___, and ___$[\vec{x}]_B$___ is the coordinate matrix of ___$\vec{x}$___ relative to ___B___. Multiplication by the transition matrix P changes a coordinate matrix relative to B' into a coordinate matrix relative to B.

Change of basis from B' to B:

$$P[\vec{x}]_{B'} = [\vec{x}]_B$$

Change of basis from B to B':

$$[\vec{x}]_{B'} = P^{-1}[\vec{x}]_B$$

The change of basis problem in example 3 can be represented by the matrix equation:

$$P = \begin{bmatrix} -6 & 4 \\ 7 & -3 \end{bmatrix}, \quad [\vec{x}]_B = \begin{bmatrix} -26 \\ 32 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-10}\begin{bmatrix} -3 & -4 \\ -7 & -6 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3/10 & 2/5 \\ 7/10 & 3/5 \end{bmatrix}$$

$$[\vec{x}]_{B'} = P^{-1}[\vec{x}]_B = \begin{bmatrix} 3/10 & 2/5 \\ 7/10 & 3/5 \end{bmatrix}\begin{bmatrix} -26 \\ 32 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

THEOREM 4.20: THE INVERSE OF A TRANSITION MATRIX

If P is the transition matrix from a basis B' to a basis B in R^n, then P is invertible and the transition matrix from B to B' is given by P^{-1}.

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ be two bases for a vector space V. If

$$\mathbf{v}_1 = c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots c_{n1}\mathbf{u}_n$$
$$\mathbf{v}_2 = c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots c_{n2}\mathbf{u}_n$$
$$\vdots$$
$$\mathbf{v}_n = c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots c_{nn}\mathbf{u}_n$$

then the transition matrix from __B__ to __B′__ is

$$Q = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

Proof (Lemma):

In text

Proof (of Theorem 4.20):

THEOREM 4.21: TRANSITION MATRIX FROM *B* TO *B′*

Let $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ and $B'=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ be two bases for R^n. Then the transition matrix __P^{-1}____ from _B___ to _B'___ can be found using Gauss-Jordan elimination on the $n\times 2n$ matrix $\begin{bmatrix}B' & B\end{bmatrix}$ as follows.

$$\begin{bmatrix}B' & B\end{bmatrix} = \begin{bmatrix}I_n & P^{-1}\end{bmatrix}$$

Example 4: Find the transition matrix from B to B'.

$B=\{(1,1),(1,0)\},\ B'=\{(1,0),(0,1)\}$

$$\begin{bmatrix}B' & B\end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad \uparrow I_2$$

$$\boxed{P^{-1} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}}$$

Example 5: Find the coordinate matrix of p relative to the standard basis for P_3.

$p = 3x^2 + 114x + 13$

Standard basis for P_3: $1x^0 + 0x^1 + 0x^2 + 0x^3,\ 0x^0 + 1x^1 + 0x^2 + 0x^3,$
$0x^0 + 0x^1 + 1x^2 + 0x^3,\ 0x^0 + 0x^1 + 0x^2 + 1x^3$

$$\{1, x, x^2, x^3\}$$

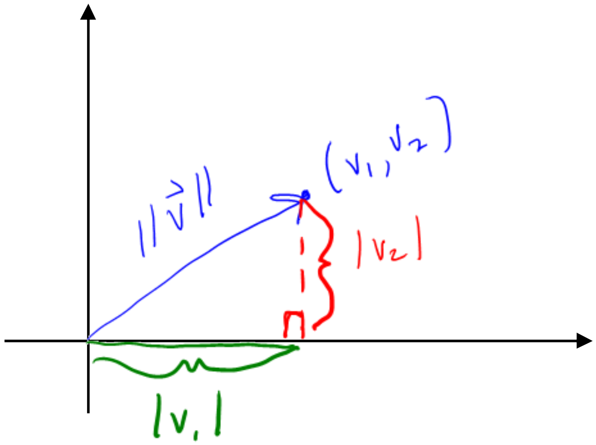
$$\vec{p} = 13(1) + 114(x) + 3(x)^2 + 0(x^3)$$

$$\boxed{[\vec{p}]_S = \begin{bmatrix} 13 \\ 114 \\ 3 \\ 0 \end{bmatrix}}$$

Section 5.1: LENGTH AND DOT PRODUCT IN R^n

When you are done with your homework you should be able to…

- Find the length of a vector and find a unit vector
- Find the distance between two vectors
- Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwartz Inequality, the triangle inequality, and the Pythagorean Theorem
- Use a matrix product to represent a dot product



$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

DEFINITION OF LENGTH OF A VECTOR IN R^n

The ___length___, or ___norm___ of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in ___R^n___ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

When would the length of a vector equal to 0?

When it's the zero vector.

Example 1: Consider the following vectors:

$$\mathbf{u} = \left(\overset{u_1,}{1}, \overset{u_2}{\frac{1}{2}}\right) \qquad \mathbf{v} = \left(\overset{v_1,}{2}, \overset{v_2}{-\frac{1}{2}}\right)$$

a. Find $\|\mathbf{u}\|$ $= \sqrt{(1)^2 + (\frac{1}{2})^2}$

$= \boxed{\frac{\sqrt{5}}{2}}$

b. Find $\|\mathbf{v}\|$ $= \sqrt{(2)^2 + (-1/2)^2}$

$= \boxed{\frac{\sqrt{17}}{2}}$

c. Find $\|\mathbf{u}+\mathbf{v}\|$ $= \|(3,0)\|$

$= \sqrt{3^2+0^2}$

$= \boxed{3}$

Find $\|\vec{u}\| + \|\vec{v}\| = \frac{\sqrt{5}}{2} + \frac{\sqrt{17}}{2}$

$= \frac{1}{2}(\sqrt{5}+\sqrt{17})$

d. Find $\|3\mathbf{u}\|$ $= \|(3, 3/2)\|$

$= \sqrt{9 + 9/4}$

$= \sqrt{\frac{45}{4}} \rightarrow = \boxed{\frac{3\sqrt{5}}{2}}$

Find $3\|\vec{u}\| = 3\left(\frac{\sqrt{5}}{2}\right) = \frac{3\sqrt{5}}{2}$

e. Any observations? $\|\vec{u}+\vec{v}\| \neq \|\vec{u}\| + \|\vec{v}\|$

$\|c\vec{u}\| = |c|\,\|\vec{u}\|$

THEOREM 5.1: LENGTH OF A SCALAR MULTIPLE

Let **v** be a vector in R^n and let c be a scalar. Then

$$\|c\vec{v}\| = |c|\|\vec{v}\|$$

where ___$|c|$___ is the ___absolute___ ___value___ of c.

Proof:

$$c\vec{v} = c(v_1, v_2, \ldots, v_n)$$

$$c\vec{v} = (cv_1, cv_2, \ldots, cv_n)$$

$$\|c\vec{v}\| = \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$\|c\vec{v}\| = \sqrt{c^2v_1^2 + c^2v_2^2 + \cdots + c^2v_n^2}$$

$$\|c\vec{v}\| = \sqrt{c^2(v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\|c\vec{v}\| = |c|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\|c\vec{v}\| = |c|\|\vec{v}\| \quad //$$

THEOREM 5.2: UNIT VECTOR IN THE DIRECTION OF v

If **v** is a nonzero vector in R^n, then the vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

has length ___1___ and has the same ___direction___ as **v**.

Proof: We know that $\vec{v} \neq \vec{0}$. So $\dfrac{1}{\|\vec{v}\|}$ is defined and positive, and $\vec{u}$ can be written as a positive scalar of $\vec{v}$.

$$\underbrace{\vec{u} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}}_{\vec{u} \text{ has the same direction as } \vec{v}} \quad \text{and} \quad \underbrace{\|\vec{u}\| = \left\|\frac{\vec{v}}{\|\vec{v}\|}\right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1.}_{\vec{u} \text{ has length } 1} \quad //$$

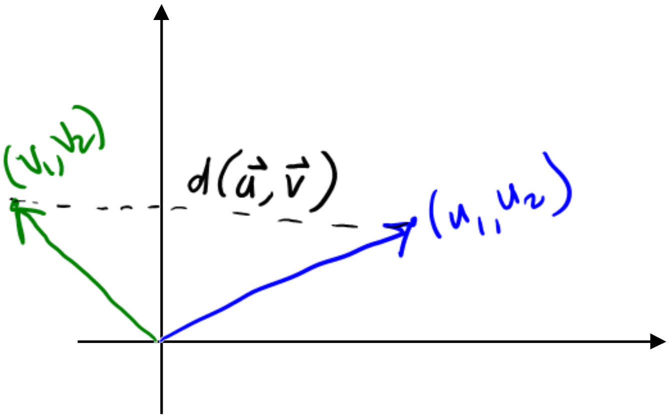
Example 2: Find the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 3$ and the same direction as $\mathbf{u} = (0, 2, 1, -1)$.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{(0,2,1,-1)}{\sqrt{(0)^2+(2)^2+(1)^2+(-1)^2}}$$

This is a unit vector in the direction of $\vec{u}$.

$$= \frac{1}{\sqrt{6}}(0,2,1,-1)$$

$$\text{So } \vec{v} = 3\frac{\vec{u}}{\|\vec{u}\|} = \frac{3}{\sqrt{6}}(0,2,1,-1) = \boxed{\frac{\sqrt{6}}{2}(0,2,1,-1)}$$



$\vec{u} = (u_1, u_2)$

$\vec{v} = (v_1, v_2)$

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$d(\vec{u},\vec{v}) = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$$

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

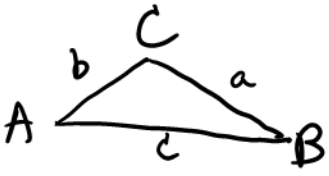
DEFINITION OF DISTANCE BETWEEN TWO VECTORS

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

Example 3: Find the distance between $\mathbf{u}=(1,1,2)$ and $\mathbf{v}=(-1,3,0)$.

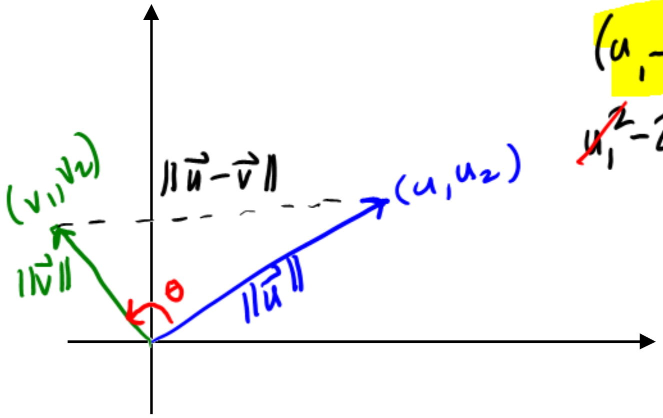
$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$
$$= \|(2,-2,2)\|$$
$$= \sqrt{(2)^2+(-2)^2+(2)^2}$$
$$= \sqrt{12}$$
$$= \boxed{2\sqrt{3}}$$



Law of Cosines: $c^2 = a^2+b^2-2ab\cos C$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$(u_1-v_1)^2+(u_2-v_2)^2 = u_1^2+u_2^2+v_1^2+v_2^2-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\cancel{u_1^2}-2u_1v_1+\cancel{v_1^2}+\cancel{u_2^2}-2u_2v_2+\cancel{v_2^2}=\cancel{u_1^2}+\cancel{u_2^2}+\cancel{v_1^2}+\cancel{v_2^2}-2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$-2u_1v_1-2u_2v_2 = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$u_1v_1+u_2v_2 = \|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\cos\theta = \frac{u_1v_1+u_2v_2}{\|\vec{u}\|\|\vec{v}\|}$$

Dot product of $\vec{u}$ and $\vec{v}$



DEFINITION OF DOT PRODUCT IN R^n

The dot product of $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ is the ___Scalar___ quantity

$$\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+\cdots+u_nv_n$$

Example 4: Consider the following vectors:

$\mathbf{u}=(-1,2)$ $\mathbf{v}=(2,-2)$

a. Find $\mathbf{u}\cdot\mathbf{v}$ $= (-1)(2)+(2)(-2)$

$= \boxed{-6}$

b. Find $\mathbf{v}\cdot\mathbf{v}$ $= (2)(2)+(-2)(-2)$

$= \boxed{8}$

$\|\vec{v}\| = \sqrt{(2)^2+(-2)^2}$

$= \sqrt{8}$

$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

c. Find $\|\mathbf{u}\|^2$ $= \vec{u}\cdot\vec{u}$

$= (-1)(-1)+(2)(2)$

$= \boxed{5}$

d. Find $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$ $= -6(2,-2)$

$= \boxed{(-12,12)}$

e. Find $\mathbf{u}\cdot(5\mathbf{v})$ $= (-1,2)\cdot(10,-10)$

$= (-1)(10)+(2)(-10)$

$= \boxed{-30}$

$5\vec{u}\cdot\vec{v} = 5(-6)$

$= \boxed{-30}$

THEOREM 5.3: PROPERTIES OF THE DOT PRODUCT

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n, and c is a scalar, then the following properties are true.

1. $\mathbf{u}\cdot\mathbf{v} =$ $\vec{v}\cdot\vec{u}$
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) =$ $\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$
3. $c(\mathbf{u}\cdot\mathbf{v}) =$ $(c\vec{u})\cdot\vec{v}$ $=$ $\vec{u}\cdot(c\vec{v})$
4. $\mathbf{v}\cdot\mathbf{v} =$ $\|\vec{v}\|^2$
5. $\mathbf{v}\cdot\mathbf{v}\geq 0,$ and $\mathbf{v}\cdot\mathbf{v}=0$ iff $\vec{v}=\vec{0}$.

Example 5: Find $(3\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-3\mathbf{v})$ given that $\mathbf{u}\cdot\mathbf{u}=8$, $\mathbf{u}\cdot\mathbf{v}=7$, and $\mathbf{v}\cdot\mathbf{v}=6$.

$= 3\vec{u}\cdot(\vec{u}-3\vec{v})-\vec{v}\cdot(\vec{u}-3\vec{v})$

$= 3\vec{u}\cdot\vec{u}-9\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+3\vec{v}\cdot\vec{v}$

$= 3(8)-9(7)-\vec{u}\cdot\vec{v}+3(6)$

$= 24-63-7+18$

$= \boxed{-28}$

THEOREM 5.4: THE CAUCHY-SCWARZ INEQUALITY

If **u** and **v** are vectors in R^n, then

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$$

where $|\vec{u}\cdot\vec{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$.

Proof:

Case 1: If $\vec{u}=\vec{0}$ then $|\vec{u}\cdot\vec{v}| = |0| = 0$ and $\|\vec{u}\|\|\vec{v}\| = 0\|\vec{v}\| = 0$. ✓

Case 2: When $\vec{u}\neq 0$, let $t\in\mathbb{R}$ and consider $t\vec{u}+\vec{v}$. Since

$(t\vec{u}+\vec{v})\cdot(t\vec{u}+\vec{v})\ge 0$, it follows that

$$t^2(\vec{u}\cdot\vec{u}) + t(\vec{u}\cdot\vec{v}) + t(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} \ge 0$$

$$t^2(\vec{u}\cdot\vec{u}) + 2t(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} \ge 0$$

Let $a=\vec{u}\cdot\vec{u}$, $b=2(\vec{u}\cdot\vec{v})$, $c=\vec{v}\cdot\vec{v}$, $at^2+bt+c\ge 0$. Since the quadratic is never negative, it either has no real roots or a single repeated root. This implies that

$$b^2-4ac\le 0$$

$$b^2\le 4ac$$

$$[2(\vec{u}\cdot\vec{v})]^2 \le 4(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v})$$

$$4(\vec{u}\cdot\vec{v})^2 \le 4(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v})$$

$$\sqrt{(\vec{u}\cdot\vec{v})^2} \le \sqrt{(\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v})}$$

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\|\vec{v}\| \quad //$$

Example 6: Verify the Cauchy-Schwarz Inequality for $\mathbf{u}=(-1,0)$ and $\mathbf{v}=(1,1)$.

$$|\vec{u}\cdot\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$$

$$|(-1,0)\cdot(1,1)| \le \sqrt{(-1)^2+(0)^2}\ \sqrt{(1)^2+(1)^2}$$

$$|-1| \le \sqrt{2}$$

$$1 \le \sqrt{2}\ \checkmark$$

DEFINITION OF THE ANGLE BETWEEN TWO VECTORS IN R^n

The ____angle____ __θ__ between two nonzero vectors in R^n is given by

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad 0 \le \theta \le \pi$$

Example 6: Find the angle between $\mathbf{u}=(2,-1)$ and $\mathbf{v}=(2,0)$.

$$\cos\theta = \frac{(2,-1)\cdot(2,0)}{\sqrt{(2)^2+(-1)^2}\sqrt{(2)^2+(0)^2}}$$

$$\cos\theta = \frac{4}{\sqrt{5}\cdot 2}$$

$$\cos\theta = \frac{2}{\sqrt{5}}$$

$$\boxed{\theta \doteq 0.4636}$$

DEFINITION OF ORTHOGONAL VECTORS

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are orthogonal if

$$\vec{u} \cdot \vec{v} = 0$$

Example 7: Determine all vectors in R^2 that are orthogonal to $\mathbf{u} = (3,1)$.

$$\vec{u} \cdot \vec{v} = 0$$

$$(3,1) \cdot (v_1, v_2) = 0$$

$$3v_1 + v_2 = 0$$

$$v_1 = -\tfrac{1}{3} v_2$$

Let $v_2 = t$

$$\boxed{\vec{v} = \left(-\tfrac{1}{3}t, t\right) = t\left(-\tfrac{1}{3}, 1\right), t \in R}$$

THEOREM 5.5: THE TRIANGLE INEQUALITY

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Proof:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \vec{u} \cdot (\vec{u} + \vec{v}) + \vec{v} \cdot (\vec{u} + \vec{v})$$

$$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

$$= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v}$$

$$= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + 2|\vec{u} \cdot \vec{v}| + \|\vec{v}\|^2$$

$$\|\vec{u} + \vec{v}\|^2 \leq \|\vec{u}\|^2 + 2|\vec{u} \cdot \vec{v}| + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$$

$$\leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

$\|\vec{u}+\vec{v}\|$ and $(\|\vec{u}\|+\|\vec{v}\|)$ are nonnegative

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\| \quad //$$

THEOREM 5.6: THE PYTHAGOREAN THEOREM

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2$$

Example 8: Verify the Pythagoren Theorem for the vectors $\mathbf{u}=(3,-2)$ and $\mathbf{v}=(4,6)$.

$$\|(3,-2)+(4,6)\|^2 = \|(3,-2)\|^2+\|(4,6)\|^2$$

$$\|(7,4)\|^2 = (\sqrt{13})^2+(\sqrt{52})^2$$

$$65 = 13+52$$

$$65 = 65 \ \checkmark$$

Section 5.2: INNER PRODUCT SPACES

When you are done with your homework you should be able to…

- π Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n, and $C[a,b]$
- π Find an orthogonal projection of a vector onto another vector in an inner product space

DEFINITION OF AN INNER PRODUCT

$c(xy)$ vs $c(x+y)$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms.

1. $\langle \mathbf{u}, \mathbf{v} \rangle =$ $\langle \vec{v}, \vec{u} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle =$ $\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
3. $c\langle \mathbf{u}, \mathbf{v} \rangle =$ $\langle c\vec{u}, \vec{v} \rangle$
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\vec{v} = \vec{0}$

$$5\left[(1,2)\cdot(3,4)\right]$$
$$= 5(11)$$
$$= 55$$
$$5\left[(1,2)\cdot(3,4)\right]$$
$$= (5,10)\cdot(3,4)$$
$$55$$

NOTE: The dot product is an example of an inner product.
$\vec{u}\cdot\vec{v}$ is the dot product (Euclidean inner product for R^n)
$\langle \vec{u}, \vec{v} \rangle$ is the general inner product for a vector space V.
A vector space V with an inner product is called an inner product space.

Example 1: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$ defines an inner product on R^3, where, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. $c, u_i, v_i, w_i \in R$

1) $\langle \vec{u}, \vec{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$

$= v_1u_1 + 2v_2u_2 + v_3u_3$ (real numbers are comm.)

$= \langle \vec{v}, \vec{u} \rangle$ ✓

2) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = u_1(v_1 + w_1) + 2u_2(v_2 + w_2) + u_3(v_3 + w_3)$

$= u_1v_1 + u_1w_1 + 2u_2v_2 + 2u_2w_2 + u_3v_3 + u_3w_3$

$= \underbrace{u_1v_1 + 2u_2v_2 + u_3v_3} + \underbrace{u_1w_1 + 2u_2w_2 + u_3w_3}$

$= \underbrace{\langle \vec{u}, \vec{v} \rangle} + \underbrace{\langle \vec{u}, \vec{w} \rangle}$ ✓

3) $c\langle \vec{u}, \vec{v} \rangle = c(u_1v_1 + 2u_2v_2 + u_3v_3)$

$= (cu_1)v_1 + 2(cu_2)v_2 + (cu_3)v_3$

$= \langle c\vec{u}, \vec{v} \rangle$

4) $\langle \vec{v}, \vec{v} \rangle = v_1v_1 + 2v_2v_2 + v_3v_3$

$= v_1^2 + 2v_2^2 + v_3^2 \geq 0$

The square of a real # ≥ 0

$\langle \vec{v}, \vec{v} \rangle = 0$

$v_1^2 + 2v_2^2 + v_3^2 = 0$ $\rightarrow$ $v_1 = v_2 = v_3 = 0$

and $\vec{v} = (0, 0, 0) = \vec{0}$ ✓

Example 2: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_2 - u_3v_3$ does not define an inner product on R^3, where , $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

Let $\vec{v} = (-1, 3, 5)$

$$\langle \vec{v}, \vec{v} \rangle = (-1)(-1) - (3)(3) - (5)(5)$$

$$= 1 - 9 - 25$$

$$= -33 < 0$$

Fails axiom 4.

THEOREM 5.7: PROPERTIES OF INNER PRODUCTS

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V , and let c be any real number.

1. $\langle \mathbf{0}, \mathbf{v} \rangle$ = $\langle \vec{v}, \vec{0} \rangle$ = 0

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle$ = $\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

Proof: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{u} + \vec{v} \rangle$ Axiom 1

$= \langle \vec{w}, \vec{u} \rangle + \langle \vec{w}, \vec{v} \rangle$ Axiom 2

$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓ Axiom 1

3. $\langle \mathbf{u}, c\mathbf{v} \rangle$ = $c\langle \vec{u}, \vec{v} \rangle$

DEFINITION OF LENGTH, DISTANCE, AND ANGLE

Let **u** and **v** be vectors in an inner product space V.

1. The length (or norm) of **u** is $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$.
2. The distance between **u** and **v** is $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$.
3. The angle between and two vectors **u** and **v** is given by

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}, \quad 0 \leq \theta \leq \pi$$

4. **u** and **v** are orthogonal when $\langle \vec{u}, \vec{v} \rangle = 0$.

If $\|\vec{u}\| = 1$, then **u** is called a unit vector. Moreover, if **v** is any nonzero vector in an inner product space V, then the vector

$u = \dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector and is called the unit vector in the direction of **v**.

$$-1 \leq \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

Inner product on $C[a,b]$ is $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$.

Inner product on $M_{2,2}$ is $\langle A,B\rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$.

Inner product on P_n is $\langle p\cdot q\rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$, where $p = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $q = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$.

Example 3: Consider the following inner product defined on R^n:

$\mathbf{u} = (0,-6)$, $\mathbf{v} = (-1,1)$, and $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + 2u_2v_2$

$$\|\vec{u}\| = \sqrt{\langle \vec{u},\vec{u}\rangle} = \sqrt{(u_1)(u_1) + 2(u_2)(u_2)}$$

a. Find $\langle \mathbf{u},\mathbf{v}\rangle$ $= \langle (0,-6),(-1,1)\rangle$

$= (0)(-1) + 2(-6)(1)$

$= \boxed{-12}$

b. Find $\|\mathbf{u}\|$ $= \sqrt{\langle (0,-6),(0,-6)\rangle}$

$= \sqrt{(0)(0) + 2(-6)(-6)}$

$= \sqrt{72}$

$= \boxed{6\sqrt{2}}$

c. Find $\|\mathbf{v}\|$ $= \sqrt{\langle \vec{v},\vec{v}\rangle}$

$= \sqrt{\langle (-1,1),(-1,1)\rangle}$

$= \sqrt{(-1)(-1) + 2(1)(1)}$

$= \boxed{\sqrt{3}}$

d. Find $d(\mathbf{u},\mathbf{v})$ $= \|\vec{u} - \vec{v}\|$

$= \|(0,-6) - (-1,1)\|$

$= \|(1,-7)\|$

$= \sqrt{\langle (1,-7),(1,-7)\rangle}$

$= \sqrt{(1)(1) + 2(-7)(-7)}$

$= \sqrt{99}$

$= \boxed{3\sqrt{11}}$

Example 4: Consider the following inner product defined:

$\langle f,g\rangle=\int_{-1}^{1} f(x)g(x)dx$, $f(x)=-x$, $g(x)=x^2-x+2$

a. Find $\langle f,g\rangle$ $=\int_a^b f(x)g(x)$

$$=\int_{-1}^{1}(-x)(x^2-x+2)dx$$

$$=\int_{-1}^{1}(-x^3+x^2-2x)dx$$

$$=\left(-\frac{1}{4}x^4+\frac{1}{3}x^3-x^2\right|_{x=-1}^{x=1}$$

$$=\left(-\cancel{\frac{1}{4}}+\frac{1}{3}-\cancel{1}\right)-\left(-\cancel{\frac{1}{4}}-\frac{1}{3}-\cancel{1}\right)=\boxed{\frac{2}{3}}$$

b. Find $\|f\|$ $=\sqrt{\langle f,f\rangle}$

$$=\sqrt{\int_a^b f(x)f(x)dx}$$

$$=\sqrt{\int_{-1}^{1}(-x)(-x)dx}$$

$$=\sqrt{\int_{-1}^{1}x^2dx}$$

$$=\sqrt{\frac{1}{3}x^3\Big|_{x=1}^{x=1}}$$

$$=\sqrt{\frac{1}{3}-\left(-\frac{1}{3}\right)}$$

$$=\boxed{\sqrt{\frac{2}{3}}\quad\text{or}\quad\frac{\sqrt{6}}{3}}$$

c. Find $\|g\| = \sqrt{\langle g,g \rangle}$

$$= \sqrt{\int_a^b g(x)g(x)\,dx}$$

$$= \sqrt{\int_{-1}^{1} (x^2 - x + 2)^2\,dx}$$

$$= \sqrt{\int_{-1}^{1} (x^4 - x^3 + 2x^2 - x^3 + x^2 - 2x + 2x^2 - 2x + 4)\,dx}$$

$$= \sqrt{\int_{-1}^{1} (x^4 - 2x^3 + 5x^2 - 4x + 4)\,dx}$$

$$= \sqrt{\left(\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{5}{3}x^3 - 2x^2 + 4x\ \right|_{x=-1}^{x=1}} \rightarrow = \sqrt{\left(\frac{1}{5} - \cancel{\frac{1}{2}} + \frac{5}{3} - \cancel{2} + 4\right) - \left(-\frac{1}{5} - \cancel{\frac{1}{2}} - \frac{5}{3} - \cancel{2} - 4\right)}$$

$$= \sqrt{\frac{2}{5} + \frac{10}{3} + 8}$$

$$= \sqrt{\frac{176}{15}}$$

$$= \boxed{\frac{4\sqrt{11}}{\sqrt{15}}}$$

d. Find $d(f,g) = \|f - g\|$

$$= \|-x^2 - 2\| \rightarrow \|(-1)(x^2 + 2)\|$$

$$= \sqrt{\int_{-1}^{1} (x^2 + 2)^2\,dx}$$

$$= \sqrt{\int_{-1}^{1} (x^4 + 4x^2 + 4)\,dx}$$

$$= \sqrt{\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\ \Big|_{x=-1}^{x=1}}$$

$$= \sqrt{\left(\frac{1}{5} + \frac{4}{3} + 4\right) - \left(-\frac{1}{5} - \frac{4}{3} - 4\right)}$$

$$= \frac{2}{5} + \frac{8}{3} + 8$$

$$= \sqrt{\frac{166}{15}}$$

Properties of length	Properties of distance
1) $\|\vec{u}\| \geq 0$	1) $d(\vec{u}, \vec{v}) \geq 0$
2) $\|\vec{u}\| = 0$ iff $\vec{u} = 0$	2) $d(\vec{u}, \vec{v}) = 0$ iff $\vec{u} = \vec{v}$
3) $\|c\vec{u}\| = \lvert c \rvert \|\vec{u}\|$	3) $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$

THEOREM 5.8

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. Cauchy-Schwarz Inequality: $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$
2. Triangle Inequality: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$
3. Pythagorean Theorem: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

   $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Example 5: Verify the triangle inequality for $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, and $\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$.

DEFINITION OF ORTHOGONAL PROJECTION

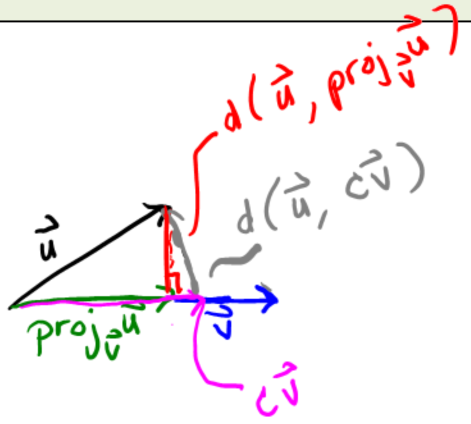
Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

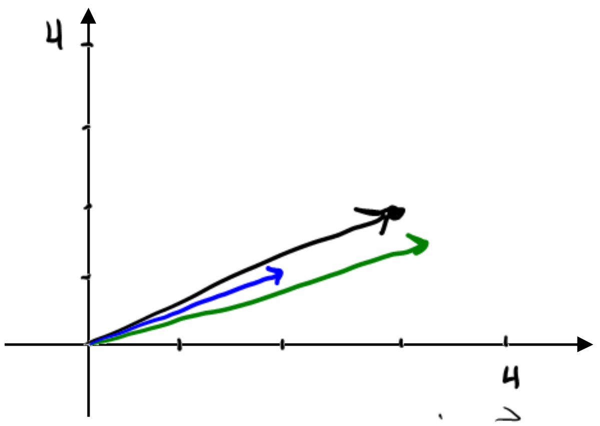
$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

THEOREM 5.9: ORTHOGONAL PROJECTION AND DISTANCE

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\vec{u}, \operatorname{proj}_{\vec{v}} \vec{u}) < d(\vec{u}, c\vec{v}), \quad c \neq \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}$$





$\vec{u} = (3,2)$

$\vec{v} = (2,1)$

$$\underline{\operatorname{proj}_{\vec{v}} \vec{u}} = \frac{\langle (3,2), (2,1) \rangle}{\langle (2,1), (2,1) \rangle} (2,1)$$

$$= \frac{8}{5}(2,1)$$

$$= \left(\frac{16}{5}, \frac{8}{5}\right) \longrightarrow = \underline{(3.2, 1.6)}$$

Example 6: Consider the vectors

dot product

$\mathbf{u} = (-1,-2)$ and $\mathbf{v} = (4,2)$. Use the Euclidean inner product to find the following:

a. $\text{proj}_{\mathbf{v}}\mathbf{u}$ $= \dfrac{\langle \vec{u},\vec{v} \rangle}{\langle \vec{v},\vec{v} \rangle}\vec{v}$

$$= \frac{(-1,-2)\cdot(4,2)}{(4,2)\cdot(4,2)}(4,2) \rightarrow$$

$$= \frac{-8}{20}(4,2)$$

$$= \left(-\frac{8}{5}, -\frac{4}{5}\right)$$

$$= (-1.6,-0.8)$$

b. $\text{proj}_{\mathbf{u}}\mathbf{v}$ $= \dfrac{\langle \vec{v},\vec{u} \rangle}{\langle \vec{u},\vec{u} \rangle}\vec{u}$

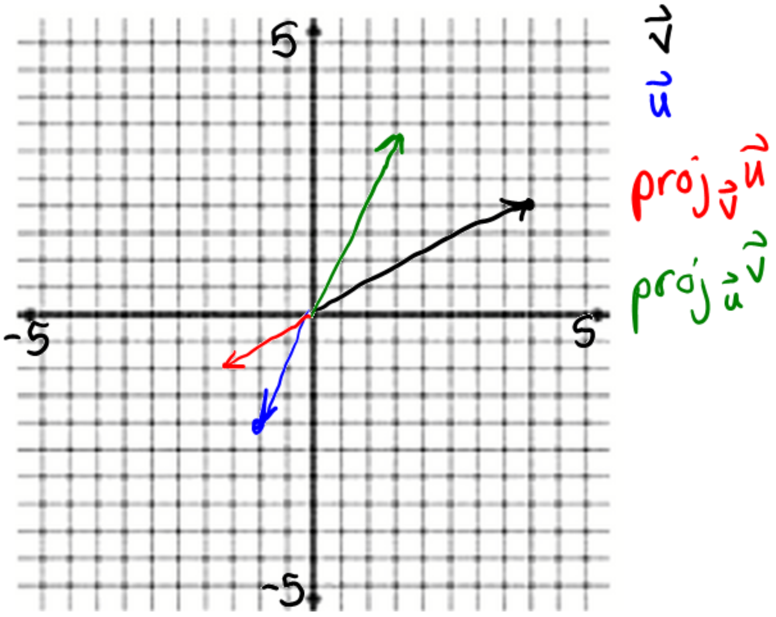
$$= \frac{-8}{(-1,-2)(-1,-2)}(-1,-2)$$

$$= -\frac{8}{5}(-1,-2) \rightarrow$$

$$= (8/5, 16/5)$$

$$= (1.6, 3.2)$$

c. Sketch the graph of both $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}$.



Section 5.3: ORTHONORMAL BASES: GRAM-SCHMIDT PROCESS

When you are done with your homework you should be able to…

- Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis
- Apply the Gram-Schmidt orthonormalization process

Consider the standard basis for R^3, which is

$$S = \{(1,0,0),(0,1,0),(0,0,1)\}$$

$$(1,0,0)\cdot(0,1,0) = 0$$
$$(1,0,0)\cdot(0,0,1) = 0$$
$$(0,1,0)\cdot(0,0,1) = 0$$

This set is the standard basis because it has useful characteristics such as…

The three vectors in the basis are unit vectors, and they are each mutually orthogonal.

DEFINITIONS OF ORTHOGONAL AND ORTHONORMAL SETS

A set S of a vector space V is called orthogonal when every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector, then S is called orthonormal. For $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, this definition has the following form.

ORTHOGONAL	**ORTHONORMAL**
1. $\langle \vec{v}_i, \vec{v}_j \rangle = 0, i \neq j$	1. $\langle \vec{v}_i, \vec{v}_j \rangle, i \neq j$
	2. $\|\vec{v}_i\| = 1, i = 1,2,3,\ldots,n$

If S is a basis, then it is an orthogonal basis or an orthonormal basis, respectively.

THEOREM 5.10: ORTHOGONAL SETS ARE LINEARLY INDEPENDENT

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of __nonzero______________ vectors in an inner product space V, then S is linearly independent.

Proof: We need to show that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$

$$\langle (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n), \vec{v}_i \rangle = \langle 0, \vec{v}_i \rangle$$

$$c_1\langle \vec{v}_1, \vec{v}_i\rangle + c_2\langle \vec{v}_2, \vec{v}_i\rangle + \cdots + c_i\langle \vec{v}_i, \vec{v}_i\rangle + \cdots + c_n\langle \vec{v}_n, \vec{v}_i\rangle = 0$$

$$c_1(0) + c_2(0) + \cdots + c_i\langle \vec{v}_i, \vec{v}_i\rangle + \cdots + c_n(0) = 0$$

$$c_i\langle \vec{v}_i, \vec{v}_i\rangle = 0$$

$$c_i\|\vec{v}_i\|^2 = 0$$

$$\|\vec{v}_i\| \neq 0$$

so $c_i = 0$.

$\therefore$ every $c_i = 0$, and the set S is linearly independent. //

THEOREM 5.10: COROLLARY

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Example 1: Consider the following set in R^4.

$$\left\{\left(\frac{\sqrt{10}}{10},0,0,\frac{3\sqrt{10}}{10}\right),(0,0,1,0),(0,1,0,0),\left(-\frac{3\sqrt{10}}{10},0,0,\frac{\sqrt{10}}{10}\right)\right\}$$

a. Determine whether the set of vectors is orthogonal.

$(\sqrt{10}/10, 0, 0, 3\sqrt{10}/10) \cdot (0,0,1,0) = 0$

$(\sqrt{10}/10, 0, 0, 3\sqrt{10}/10) \cdot (0,1,0,0) = 0$

$(\sqrt{10}/10, 0, 0, 3\sqrt{10}/10) \cdot (-3\sqrt{10}/10, 0, 0, \sqrt{10}/10) = 0$

$(0,0,1,0) \cdot (0,1,0,0) = 0$

$(0,0,1,0) \cdot (-3\sqrt{10}/10, 0, 0, \sqrt{10}/10) = 0$

$(0,1,0,0) \cdot (-3\sqrt{10}/10, 0, 0, \sqrt{10}/10) = 0$

This set is orthogonal since its vectors are mutually orthogonal.

b. If the set is orthogonal, then determine whether it is also orthonormal.

$\|(\sqrt{10}/10, 0, 0, 3\sqrt{10}/10)\| = 1$

$\|(0,0,1,0)\| = 1$

$\|(0,1,0,0)\| = 1$

$\|(-3\sqrt{10}/10, 0, 0, \sqrt{10}/10)\| = 1$

yes, since the set is orthogonal and each vector has a length of 1.

c. Determine whether the set is a basis for R^4.

Yes since the set has 4 vectors and it's orthogonal.

THEOREM 5.11: COORDINATES RELATIVE TO AN ORTHONORMAL BASIS

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ relative to B is

$$\vec{w} = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{w}, \vec{v}_n \rangle \vec{v}_n$$

Proof: Since B is a basis for the inner product space V, $\exists$ unique scalars $c_1, c_2, \ldots, c_n \ni \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

$$\langle \vec{w}, \vec{v}_i \rangle = \langle (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n), \vec{v}_i \rangle$$

$$\langle \vec{w}, \vec{v}_i \rangle = c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \cdots + c_i \langle v_i, v_i \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_i \rangle$$

$$\langle \vec{w}, \vec{v}_i \rangle = c_1(0) + c_2(0) + \cdots + c_i \langle \vec{v}_i, \vec{v}_i \rangle + \cdots + c_n(0) \quad B \text{ is orthogonal}$$

$$\langle \vec{w}, \vec{v}_i \rangle = c_i \langle v_i, v_i \rangle$$

$$\langle w, \vec{v}_i \rangle = c_i \|v_i\|^2$$

$$\langle \vec{w}, \vec{v}_i \rangle = c_i (1)^2 = c_i \quad B \text{ is orthonormal} \quad //$$

The coordinates of ___$\vec{w}$___ relative to the ___orthonormal___ basis ___B___ are called the ___Fourier___ coefficients of ___$\vec{w}$___ relative to ___B___. The corresponding coordinate matrix of ___$\vec{w}$___ relative to ___B___ is

$$[\vec{w}]_B = [c_1 \; c_2 \; c_3 \; \cdots \; c_n]^T$$

$$= [\langle \vec{w}, \vec{v}_1 \rangle \; \langle \vec{w}, \vec{v}_2 \rangle \cdots \langle \vec{w}, \vec{v}_n \rangle]^T$$

Example 2: Show that the set of vectors $\{(2,-5),(10,4)\}$ in R^2 is orthogonal and normalize the set to produce an orthonormal set.

$(2,-5)\cdot(10,4) = 20-20 = 0$ so the set is orthogonal.

$$\frac{(2,-5)}{\|(2,-5)\|} = \frac{(2,-5)}{\sqrt{29}}$$

$$\boxed{\left\{\left(\tfrac{2}{\sqrt{29}}, \tfrac{-5}{\sqrt{29}}\right), \left(\tfrac{5}{\sqrt{29}}, \tfrac{2}{\sqrt{29}}\right)\right\}}$$

$$\frac{(10,4)}{\|(10,4)\|} = \frac{(10,4)}{\sqrt{116}} = \frac{(10,4)}{2\sqrt{29}}$$

Example 3: Find the coordinate matrix of $\mathbf{x} = (-3, 4)$ relative to the orthonormal basis $B = \left\{ \left(\underset{\vec{v}_1}{\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}} \right), \left(\underset{\vec{v}_2}{-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}} \right) \right\}$ in R^2.

$$[\vec{x}]_B = \left[\langle \vec{x}, \vec{v}_1 \rangle \quad \langle \vec{x}, \vec{v}_2 \rangle \right]^T$$

$$\langle \vec{x}, \vec{v}_1 \rangle = (-3,4) \cdot \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right) = \sqrt{5}$$

$$\langle \vec{x}, \vec{v}_2 \rangle = (-3,4) \cdot \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) = 2\sqrt{5}$$

$$\boxed{[\vec{x}]_B = \left[\sqrt{5} \quad 2\sqrt{5} \right]^T}$$

THEOREM 5.12: GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

1. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product V .
2. Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where $\mathbf{w}_i$ is given by

$$\mathbf{w}_1 = \mathbf{v}_1$$

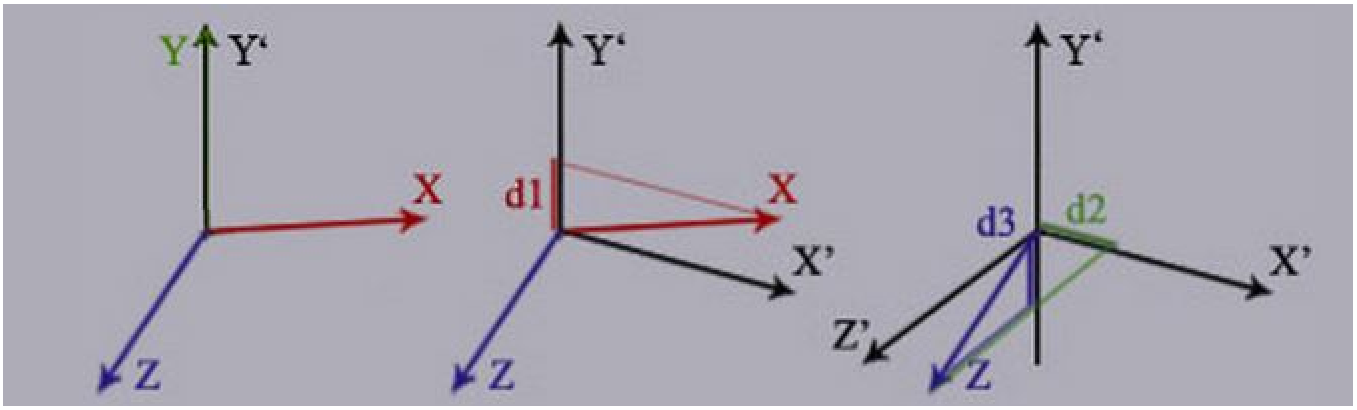
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle}\mathbf{w}_{n-1}$$

3. Let $\mathbf{u}_i = \dfrac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for V . Moreover, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.



Example 4: Apply the Gram-Schmidt orthonormalization process to transform the basis $B=\{(1,0,0),(1,1,1),(1,1,-1)\}$ for a subspace in R^3 into an orthonormal basis. Use the Euclidean inner product on R^3 and use the vectors in the order they are given.

$$\vec{w}_1 = \vec{v}_1 = (1,0,0)$$

$$\vec{w}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{w}_1 \rangle \frac{\vec{w}_1}{\langle \vec{w}_1, \vec{w}_1 \rangle} = (1,1,1) - (1,1,1)\cdot(1,0,0)\left[\frac{(1,0,0)}{(1,0,0)\cdot(1,0,0)}\right]$$

$$\vec{w}_2 = (1,1,1) - (1)\frac{(1,0,0)}{1} = (0,1,1)$$

$$\vec{w}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{w}_1 \rangle \frac{\vec{w}_1}{\langle w_1, w_1 \rangle} - \langle \vec{v}_3, \vec{w}_2 \rangle \frac{\vec{w}_2}{\langle \vec{w}_2, \vec{w}_2 \rangle}$$

$$\vec{w}_3 = (1,1,-1) - (1,1,-1)\cdot(1,0,0)\left[\frac{(1,0,0)}{(1,0,0)\cdot(1,0,0)}\right] - (1,1,-1)\cdot(0,1,1)\left[\frac{(0,1,1)}{(0,1,1)\cdot(0,1,1)}\right]$$

$$\vec{w}_3 = (1,1,-1) - (1)\frac{(1,0,0)}{1} - 0\left[\frac{(0,1,1)}{2}\right]$$

$$\vec{w}_3 = (0,1,-1)$$

$$B' = \{\underset{\vec{w}_1}{(1,0,0)}, \underset{\vec{w}_2}{(0,1,1)}, \underset{\vec{w}_3}{(0,1,-1)}\}$$ B′ is an orthogonal set

$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{(1,0,0)}{1} = (1,0,0), \quad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{(0,1,1)}{\sqrt{2}} = (0, 1/\sqrt{2}, 1/\sqrt{2})$$

$$u_3 = \frac{\vec{w}_3}{\|w_3\|} = \frac{(0,1,-1)}{\sqrt{2}} = (0, 1/\sqrt{2}, -1/\sqrt{2})$$

$$B'' = \{\underset{\vec{u}_1}{(1,0,0)}, \underset{\vec{u}_2}{(0, 1/\sqrt{2}, 1/\sqrt{2})}, \underset{\vec{u}_3}{(0, 1/\sqrt{2}, -1/\sqrt{2})}\}$$

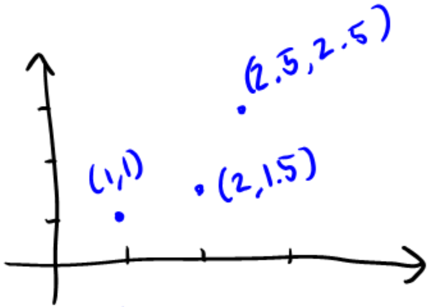
B″ is orthonormal

Section 5.4: MATHEMATICAL MODELS AND LEAST SQUARES ANALYSIS

When you are done with your homework you should be able to…

- π Define the least squares problem
- π Find the orthogonal complement of a subspace and the projection of a vector onto a subspace
- π Find the four fundamental subspaces of a matrix
- π Solve a least squares problem *
- π Use least squares for mathematical modeling

In this section we will study inconsistent systems of linear equations and learn how to find the best possible solution of such a system.



$$\underline{c_0 + c_1 x = y}$$

$$c_0 + 1c_1 = 1$$

$$c_0 + 2c_1 = 1.5$$

$$c_0 + 2.5c_1 = 2.5$$

norm of the error:

$$\|A\vec{c} - \vec{b}\|$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2.5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 1.5 \\ 2.5 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

LEAST SQUARES PROBLEM

Given an $m \times n$ matrix A and a vector $\mathbf{b}$ in R^m, the least squares problem is to find $\vec{x}$ in R^m such that $\|A\vec{x} - \vec{b}\|^2$ is minimized.

DEFINITION OF ORTHOGONAL SUBSPACES

The subspaces S_1 and S_2 of R^n are orthogonal when $\vec{v}_1 \cdot \vec{v}_2 = 0$ for all $\mathbf{v}_1$ in S_1 and $\mathbf{v}_2$ in S_2.

Example 1: Are the following subspaces orthogonal?

$$S_1 = \text{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \text{and} \quad S_2 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

1st space

2nd space

$(0,-1,1)\cdot(0,1,1) = 0$

$(1,0,0)\cdot(0,1,1) = 0$

$\boxed{\text{So, } S_1 \text{ and } S_2 \text{ are orthogonal.}}$

DEFINITION OF ORTHOGONAL COMPLEMENT

If S is a subspace of R^n, then the orthogonal complement of S is the set

$$S^{\perp} = \{\vec{u} \in R^n : \vec{v}\cdot\vec{u} = 0 \text{ for all vectors } \vec{v} \in S\}$$

What's the orthogonal complement of $\{\mathbf{0}\}$ in R^n?

All of R^n.

What's the orthogonal complement of R^n?

$\{\vec{0}\}$

DEFINITION OF DIRECT SUM

Let S_1 and S_2 be two subspaces of R^n. If each vector $\vec{x} \in R^n$ can be uniquely written as the sum of a vector $\vec{s}_1$ from S_1 and a vector $\vec{s}_2$ from S_2, $\vec{x} = \vec{s}_1 + \vec{s}_2$, then R is the direct sum of S_1 and S_2, and you can write $R^n = S_1 \oplus S_2$.

Example 2: Find the orthogonal complement $S^{\perp}$, and find the direct sum $S \oplus S^{\perp}$.

$$S = \text{span}\left\{\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right\}$$

↑

S_1

consider the transpose: $(0,1,-1,1)$

$(1,0,0,0), (0,1,1,0), (0,0,1,1)$

$$\boxed{S^{\perp} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}}$$

↑

S_2

$$\boxed{S \oplus S^{\perp} = R^4}$$

THEOREM 5.13: PROPERTIES OF ORTHOGONAL SUBSPACES

Let S be a subspace of R^n, Then the following properties are true.

1. $\dim(S) + \dim(S^{\perp}) = n$
2. $R^n = S \oplus S^{\perp}$
3. $(S^{\perp})^{\perp} = S$

THEOREM 5.14: PROJECTION ONTO A SUBSPACE

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ is an orthonormal basis for the subspace S of R^n, and $\mathbf{v} \in R^n$, then

$$\text{proj}_S \vec{v} = (\vec{v}\cdot\vec{u}_1)(\vec{u}_1) + (\vec{v}\cdot\vec{u}_2)(\vec{u}_2) + \cdots + (\vec{v}\cdot\vec{u}_t)(\vec{u}_t)$$

Example 3: Find the projection of the vector $\mathbf{v}$ onto the subspace S.

$$S = \text{span}\left\{ \underset{\vec{w}_1}{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{w}_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{w}_3}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

S is orthogonal but not normal.

$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{(-1,2,0,0)^T}{\sqrt{5}} = (-1/\sqrt{5}, 2/\sqrt{5}, 0, 0)^T$$

$$\vec{u}_2 = \vec{w}_2$$

$$\vec{u}_3 = \vec{w}_3$$

$$\begin{aligned} \text{proj}_S \vec{v} &= (1,1,1,1)\cdot(-1/\sqrt{5}, 2/\sqrt{5}, 0, 0)(-1/\sqrt{5}, 2/\sqrt{5}, 0, 0) \\ &+ (1,1,1,1)\cdot(0,0,1,0)(0,0,1,0) \\ &+ (1,1,1,1)\cdot(0,0,0,1)(0,0,0,1) \\ &= \frac{1}{\sqrt{5}}(-1/\sqrt{5}, 2/\sqrt{5}, 0, 0) + 1(0,0,1,0) + (1)(0,0,0,1) \\ &= \left(-\frac{1}{5}, \frac{2}{5}, 0, 0\right) + (0,0,1,0) + (0,0,0,1) \\ &= \boxed{\left(-\frac{1}{5}, \frac{2}{5}, 1, 1\right)^T} \end{aligned}$$

THEOREM 5.15: ORTHOGONAL PROJECTION AND DISTANCE

Let S be a subspace of R^n and let $\mathbf{v} \in R^n$. Then, for all $\mathbf{u} \in S$, $\mathbf{u} \neq \text{proj}_S \mathbf{v}$,

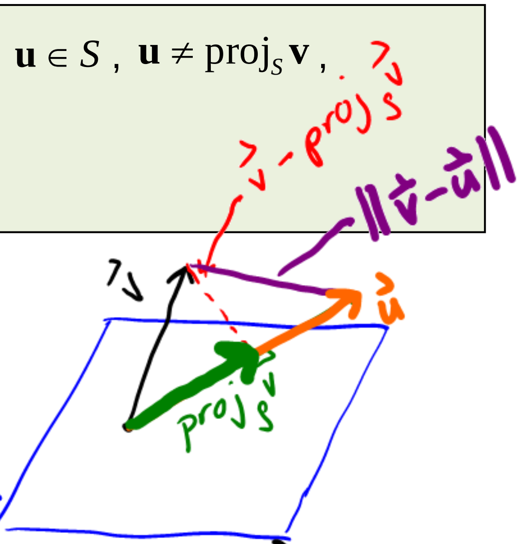
$$\|\vec{v} - \text{proj}_S \vec{v}\| < \|\vec{v} - \vec{u}\|$$

Proof: Let $\vec{u} \in S$, $\vec{u} \neq \text{proj}_S \vec{v}$.

$$\vec{v} - \vec{u} = (\vec{v} - \text{proj}_S \vec{v}) + (\text{proj}_S \vec{v} - \vec{u})$$

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{v} - \text{proj}_S \vec{v}\|^2 + \|\text{proj}_S \vec{v} - \vec{u}\|^2$$

$$\|\vec{v} - \vec{u}\| > \|\vec{v} - \text{proj}_S \vec{v}\| \quad //$$



* $\text{proj}_S \vec{v} - \vec{u} \in S$ and $\vec{v} - \text{proj}_S \vec{v} \perp S$ so you can use the pythagorean theorem.

* $\vec{u} \neq \text{proj}_S \vec{v}$, so $\|\text{proj}_S \vec{v} - \vec{u}\| > 0$

FUNDAMENTAL SUBSPACES OF A MATRIX

Recall that if A is an $m \times n$ matrix, then the column space of A is a Subspace of R^m consisting of all vectors of the form $A\vec{x}$, $\vec{x} \in R^n$. The four fundamental subspaces of the matrix A are defined as follows.

$N(A)$ = nullspace of A $\qquad$ $N(A^T)$ = nullspace of A^T

$R(A)$ = column space of A $\qquad$ $R(A^T)$ = column space of A^T

Example 4: Find bases for the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \xrightarrow{\text{rref}} \underline{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Column space of A:

$$R(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \underline{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Column Space of A^T:

$$R(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Null space of A:

$$\underline{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$

let $x_3 = t, t \in \mathbb{R}$

$$x_1 = -2x_3 = -2t$$
$$x_2 = x_3 = t$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

Null space for A^T:

$$\underline{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 \rightarrow x_1 = -x_3 = -t$$
$$x_2 + x_3 = 0 \rightarrow x_2 = -x_3 = -t$$

let $x_3 = t, t \in \mathbb{R}$

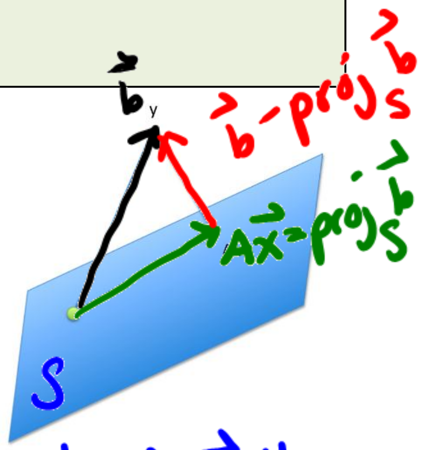
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{N(A^T) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

THEOREM 5.16: FUNDAMENTAL SUBSPACES OF A MATRIX

If A is an $m \times n$ matrix, then

1. $R(A)$ and $N(A^T)$ are orthogonal subspaces of R^m.
2. $R(A^T)$ and $N(A)$ are orthogonal subspaces of R^n.
3. $R(A) \oplus N(A^T) = R^m$
4. $R(A^T) \oplus N(A) = R^n$



SOLVING THE LEAST SQUARES PROBLEM

Recall that we are attempting to find a vector $\mathbf{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$, where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector in R^m. Let S be the column space of A: $S = R(A)$. Assume that $\mathbf{b}$ is not in S, because otherwise the system $A\mathbf{x} = \mathbf{b}$ would be consistent. We are looking for a vector $A\vec{x}$ in S that is as close as possible to $\vec{b}$. This desired vector is the projection of $\vec{b}$ onto S. So, $A\vec{x} = \text{proj}_S \vec{b}$ and $A\vec{x} - \vec{b}$ = $\text{proj}_S \vec{b} - \vec{b}$ is orthogonal to $S = R(A)$. However, this implies that $A\vec{x} - \vec{b}$ is in $R(A^{\perp})$, which equals $N(A^T)$. So, $A\vec{x} - \vec{b}$ is in the nullspace of A^T.

$$A^T(A\vec{x}-\vec{b})=\vec{0}$$
$$A^TA\vec{x}-A^T\vec{b}=\vec{0}$$
$$A^TA\vec{x}=A^T\vec{b}$$

The solution of the least squares problem comes down to solving the _nxn______ linear system of equations _$A^TA\vec{x}=A^T\vec{b}$______. These equations are called the _normal_______ equations of the least squares problem _$A\vec{x}=\vec{b}$______.

Example 5: Find the least squares solution of the system $A\mathbf{x}=\mathbf{b}$.

$$A=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}2\\1\\0\\2\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & 1 & 0 & 1\\ -1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}$$

$$A^TA\vec{x}=A^T\vec{b}$$

$$\underset{3\times4}{\begin{bmatrix}1 & 1 & 0 & 1\\ -1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}}\underset{4\times3}{\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underset{3\times4}{\begin{bmatrix}1 & 1 & 0 & 1\\ -1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{bmatrix}}\underset{4\times1}{\begin{bmatrix}2\\1\\0\\2\end{bmatrix}}$$

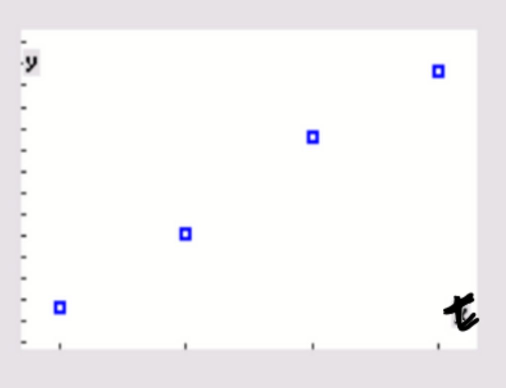
$$\underset{3\times3}{\begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 1\\ 3 & 1 & 4\end{bmatrix}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underset{3\times1}{\begin{bmatrix}5\\-1\\5\end{bmatrix}}$$

$$\left[\begin{array}{ccc|c}3 & 0 & 3 & 5\\ 0 & 3 & 1 & -1\\ 3 & 1 & 4 & 5\end{array}\right]=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 7/6\\ 0 & 1 & 0 & -1/2\\ 0 & 0 & 1 & 1/2\end{array}\right]=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\rightarrow \begin{array}{l}x_1=7/6\\ x_2=-1/2\\ x_3=1/2\end{array}\rightarrow \boxed{\vec{x}=\begin{bmatrix}7/6\\-1/2\\1/2\end{bmatrix}}$$

Example 6: The table shows the numbers of doctoral degrees *y* (in thousands) awarded in the United States from 2005 through 2008. Find the least squares regression line for the data. Then use the model to predict the number of degrees awarded in 2015. Let *t* represent the year, with *t* = 5 corresponding to 2005. (Source: U.S. National Center for Education Statistics)

Year	2005 5	2006 6	2007 7	2008 8
Doctoral Degrees, *y*	52.6	56.1	60.6	63.7



Linear Trend ✓

$$A^T A\vec{c} = A^T\vec{b}$$

$$c_0 + c_1 t = y$$

$$1c_0 + 5c_1 = 52.6$$
$$1c_0 + 6c_1 = 56.1$$
$$1c_0 + 7c_1 = 60.6$$
$$1c_0 + 8c_1 = 63.7$$

$$A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 52.6 \\ 56.1 \\ 60.6 \\ 63.7 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$$A^T A\vec{c} = A^T\vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} 52.6 \\ 56.1 \\ 60.6 \\ 63.7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 26 \\ 26 & 175 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 233.00 \\ 1533.40 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 26 & 233.0 \\ 26 & 175 & 1533.4 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 33.68 \\ 0 & 1 & 3.78 \end{bmatrix}$$

$$c_0 = 33.68$$
$$c_1 = 3.78$$

$$\hat{y}(t) = 33.68 + 3.78t$$

$$\hat{y}(15) = 90.38$$

If this linear trend continues, we predict that 90,380 doctoral degrees will be awarded in 2015.

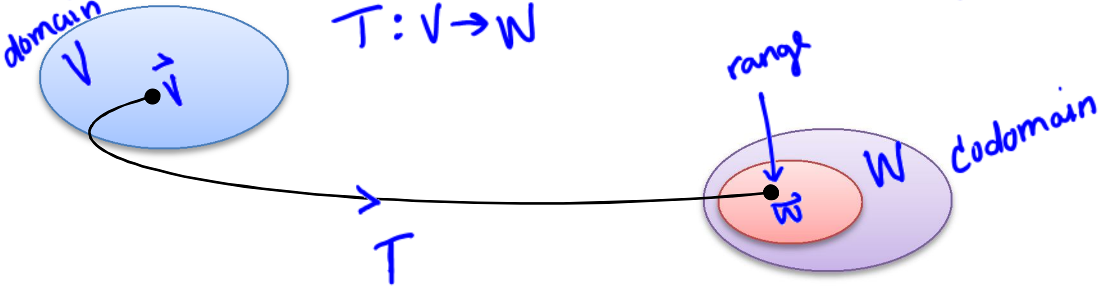
Section 6.1: INTRODUCTION TO LINEAR TRANSFORMATIONS

When you are done with your homework you should be able to…

- Find the image and preimage of a function
- Show that a function is a linear transformation, and find a linear transformation

IMAGES AND PREIMAGES OF FUNCTIONS

In this section we will learn about functions that map a vector space V onto a vector space W. This is denoted by $T: V \to W$.
The standard function terminology is used for such functions. V is called the domain of T, and W is called the codomain of T. If is $\mathbf{v}$ in V, and $\mathbf{w}$ in W such that $T(\vec{v}) = \vec{w}$, $\vec{w}$ is called the image of $\vec{v}$ under T. The set of all images of vectors in V is called the range of T, and the set of all $\mathbf{v}$ in V such that $T(\vec{v}) = \vec{w}$ is called the preimage of $\vec{w}$.



Example 1: Use the function to find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$.

part a

$$T(v_1, v_2) = (2v_2 - v_1, v_1, v_2), \quad \mathbf{v} = (\underset{v_1}{0}, \underset{v_2}{6}), \quad \mathbf{w} = (3,1,2)$$

for part b

$$T(0,6) = (2(6)-(0), 0, 6) = \boxed{(12,0,6)}$$

So (0,6) is the pre-image of (12,0,6) under $T: R^2 \to R^3$

and (12,0,6) is the image of (0,6) under T.

$T(v_1, v_2) = (3,1,2)$

$(2v_2 - v_1, v_1, v_2) = (3,1,2)$

$2v_2 - v_1 = 3 \to 2(2) - 1 = 3$ ✓

$v_1 = 1$

$v_2 = 2$

$$\boxed{T(1,2) = (3,1,2)}$$

DEFINITION OF A LINEAR TRANSFORMATION

Let V and W be vector spaces. The function $T : V \to W$ is called a linear transformation of __V__ ~~into~~ onto __W__ when the following two properties are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and any scalar c.

1. $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$

A linear transformation is __operation__ __preserving__ because the same result occurs whether you perform the operations of addition and scalar multiplication __before__ or __after__ applying the __linear__ __transformation__. Although the same symbols denote the vector operations in both V and W, you should note that the operations may be different.

Addition in V Addition in W Scalar mult in V Scalar mult in W

$T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ $T(c\vec{u}) = cT(\vec{u})$

Example 2: Determine whether the function is a linear transformation.

a. $T : R^3 \to R^3,\ T(x,y,z) = (x+1, y+1, z+1)$

$\vec{v} = (1,1,1)$

$\vec{u} = (1,2,3)$

$$T(\vec{u}+\vec{v}) \overset{?}{=} T(\vec{u})+T(\vec{v})$$

$$T[(1,1,1)+(1,2,3)] \overset{?}{=} (1+1,2+1,3+1)+(1+1,1+1,1+1)$$

$$T(2,3,4) \overset{?}{=} (2,3,4)+(2,2,2)$$

$$(2+1,3+1,4+1) \overset{?}{=} (4,5,6)$$

$$(3,4,5) \neq (4,5,6)$$

not closed under addition

$\therefore T: M_{22} \to R, T(A) = a+b+c+d$ is a linear transformation

b. $T: M_{2,2} \to R,\ T(A) = a+b+c+d$

Check it out: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = B$

$T(A+B) \overset{?}{=} T(A) + T(B)$

$T\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \overset{?}{=} (1+2+3+4) + (5+6+7+8)$

$6+8+10+12 \overset{?}{=} 10 + 26$

$36 = 36$ ✓

Let $A = [a_{ij}]$, $B = [b_{ij}]$, A, B are 2×2, $a_{ij}, b_{ij}, c \in R$

$$T(A+B) = T([a_{ij}] + [b_{ij}])$$
$$= T([a_{ij} + b_{ij}])$$
$$= (a_{11}+b_{11}) + (a_{12}+b_{12}) + (a_{21}+b_{21}) + (a_{22}+b_{22})$$
$$= (a_{11}+a_{12}+a_{21}+a_{22}) + (b_{11}+b_{12}+b_{21}+b_{22})$$
$$\to T(A) + T(B) ✓$$

$$T(cA) = T(c[a_{ij}])$$
$$= T([ca_{ij}])$$
$$= ca_{11} + ca_{12} + ca_{21} + ca_{22}$$
$$= c(a_{11} + a_{12} + a_{21} + a_{22})$$
$$= cT(A) ✓$$

THEOREM 6.1: PROPERTIES OF LINEAR TRANSFORMATIONS

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the following properties are true.

1. $T(\vec{0}) = \vec{0}$
2. $T(-\vec{v}) = -T(\vec{v})$
3. $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$

Proof: Let $\vec{u}, \vec{v} \in V$

$$T(\vec{u} - \vec{v}) = T(\vec{u} + (-\vec{v}))$$
$$= T(\vec{u}) + T(-\vec{v})$$
$$= T(\vec{u}) + [-T(\vec{v})]$$
$$= T(\vec{u}) - T(\vec{v}) \ //$$

4. If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, then $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n)$

= $T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \cdots + T(c_n\vec{v}_n)$

Example 3: Let $T: R^3 \to R^3$ be a linear transformation such that $\underline{T(1,0,0)} = (2,4,-1)$, $\underline{T(0,1,0)} = (1,3,-2)$, and $\underline{T(0,0,1)} = (0,-2,2)$. Find the indicated image.

$$T(2,-1,0) = T[(2,0,0) + (0,-1,0) + (0,0,0)]$$
$$= T[2(1,0,0) + -1(0,1,0) + 0(0,0,1)]$$
$$= T[2(1,0,0)] + T[-(0,1,0)] + T[0(0,0,1)]$$
$$= 2T(1,0,0) - T(0,1,0) + 0T(0,0,1)$$
$$= 2(2,4,-1) - (1,3,-2) + 0(0,-2,2)$$
$$= (4,8,-2) + (-1,-3,2) + (0,0,0) \longrightarrow = \boxed{(3,5,0)}$$

THEOREM 6.2: THE LINEAR TRANSFORMATION GIVEN BY A MATRIX

Let A be an $m \times n$ matrix. The function T defined by

$$T(\vec{v}) = A\vec{v}$$

is a linear transformation from R^n into R^m. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in R^n and $m \times 1$ matrices represent the vectors in R^m.

$$\mathbf{Av} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \underset{\substack{\uparrow \\ \text{vector} \\ \text{in } R^n}}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}} = \begin{bmatrix} a_{11}v_1 + & \cdots & +a_{1n}v_n \\ \vdots & \ddots & \vdots \\ \underline{a_{m1}v_1} + & \cdots & +\underline{a_{mn}v_n} \end{bmatrix}$$

vectors in R^m

Example 4: Define the linear transformation $T: R^n \to R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m.

a.

$$A=\overbrace{\begin{bmatrix}1&2\\-2&4\\-2&2\end{bmatrix}}^{2\text{ columns}}\Bigg\}3\text{ rows}$$

A has size 3x2.

$$\boxed{\begin{array}{l}\text{dimension of } R^n = 2\\ \text{dimension of } R^m = 3\end{array}}$$

b.

$$A=\overbrace{\begin{bmatrix}1&3&-1&0\\0&1&-2&1\\2&1&-4&1\end{bmatrix}}^{4\text{ columns}}\Bigg\}3\text{ rows}$$

$$\boxed{\begin{array}{l}\dim(R^n) = 4\\ \dim(R^m) = 3\end{array}}$$

Example 5: Consider the linear transformation from Example 4, part a.

a. Find $T(2,4)$

$$T(\vec{v}) = A\vec{v}$$

$$A=\begin{bmatrix}1&2\\-2&4\\-2&2\end{bmatrix}$$

$$\vec{v} = (2,4)$$

$$T(2,4) = \begin{bmatrix}1&2\\-2&4\\-2&2\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}10\\12\\4\end{bmatrix}$$

$$\boxed{T(2,4) = (10,12,4)}$$

$T(\vec{v}) = (-1,2,2)$

b. Find the preimage of $(-1,2,2)$

$T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix}1 & 2\\ -2 & 4\\ -2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}-1\\ 2\\ 2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & -1\\ -2 & 4 & 2\\ -2 & 2 & 2\end{array}\right] = \begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{array}\right] = \begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$\boxed{(-1,0) \text{ is the preimage under } T.}$

$T(-1,0) = (-1,2,2)$

c. Explain why the vector $(1,1,1)$ has no preimage under this transformation.

$T(\vec{v}) = (1,1,1)$

$$\begin{bmatrix}1 & 2\\ -2 & 4\\ -2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 2 & 1\\ -2 & 4 & 1\\ -2 & 2 & 1\end{array}\right] = \begin{bmatrix}v_1\\ v_2\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]$$

inconsistent
no solution

there doesn't exist

such that

$\boxed{\text{no preimage since } \nexists\ \vec{v} \ni T(\vec{v}) = (1,1,1)}$

Example 6: Let T be the linear transformation from P_2 into R given by the integral $T(p)=\int_0^1 p(x)dx$. Find the preimage of 1. That is, find the polynomial function(s) of degree 2 or less such that $T(p)=1$.

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$T(p) = 1$$

$$T(p) = \int_0^1 p(x)\,dx$$

$$1 = \int_0^1 (a_0 + a_1 x + a_2 x^2)\,dx$$

$$1 = \left(a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} \right|_{x=0}^{x=1}$$

$$1 = \left[\left(a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 \right) - (0+0+0) \right]$$

$$1 = a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2$$

$$a_0 = 1 - \frac{1}{2}a_1 - \frac{1}{3}a_2$$

Let $a_1 = -2a$, $a_2 = -3b$

$$a_0 = 1 + a + b$$
$$a_1 = -2a$$
$$a_2 = -3b$$

$$\boxed{p(x) = (1+a+b) + (-2a)x + (-3b)x^2 \in P_2}$$

Section 6.2: THE KERNEL AND RANGE OF A LINEAR TRANSFORMATIONS

When you are done with your homework you should be able to…

- π Find the kernel of a linear transformation
- π Find a basis for the range, the rank, and the nullity of a linear transformation
- π Determine whether a linear transformation is one-to-one or onto
- π Determine whether vector spaces are isomorphic

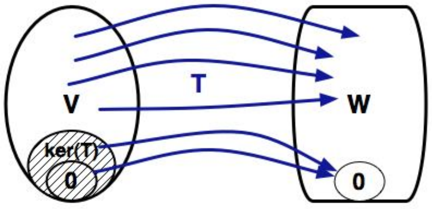
THE KERNEL OF A LINEAR TRANSFORMATION

We know from Theorem 6.1 that for any linear transformation $T: V \to W$, the zero vector in V maps to the zero vector in W. That is, $T(\vec{0}) = \vec{0}$. In this section, we will consider whether there are other vectors $\vec{v}$ such that $T(\vec{v}) = \vec{0}$. The collection of all such elements is called the Kernel of T. Note that the zero vector is denoted by the symbol $\vec{0}$ in both V and W, even though these two zero vectors are often different.

In R^2, $\vec{0} = (0,0)$

In R^3, $\vec{0} = (0,0,0)$

For 2x2 matrices, $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



ker(T) = {v ∈ V | T.v = 0}

V and W are vector spaces, T function from V to W
T is a linear transform/map = homomorphism
(means that T(x+y) = T(x)+T(y), T(a.x) = a.T(x))

DEFINITION OF KERNEL OF A LINEAR TRANSFORMATION

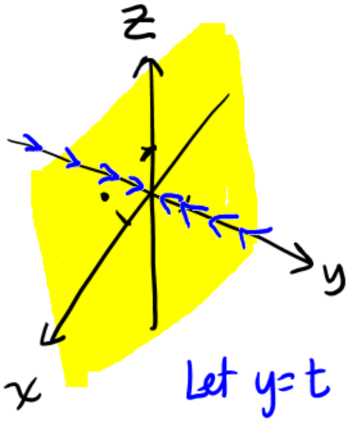
Let $T: V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\vec{v}) = \vec{0}$ is called the Kernel of T and is denoted by $\ker(T)$.

Example 1: Find the kernel of the linear transformation.

a. $T: R^3 \rightarrow R^3, T(x,y,z) = (x,0,z)$

$$\boxed{\text{Ker}(T) = \{(0,t,0) : t \in R\}}$$

Since $x=x$, $y=0$, $z=z$ under T, so $x=z=0$, and $y=t$, $t \in R$



Let $y=t$

b. $T: P_3 \rightarrow P_2, T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2$

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2 = 0$$

$$a_1 = a_2 = a_3 = 0$$

$$\text{Ker}(T) = \{a_0 : a_0 = 0\}$$

c.

$T: P_2 \rightarrow R,$

$T(p) = \int_0^1 p(x)dx$

For P_2, $p(x) = a_0 + a_1x + a_2x^2$

$$0 = \int_0^1 (a_0 + a_1x + a_2x^2)\,dx$$

$$0 = \left(a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3\right)\Big|_{x=0}^{x=1}$$

$$0 = \left[\left(a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2\right) - (0+0+0)\right]$$

$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = 0$, $a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2$

Let $a_1 = -2a$,

$a_2 = -3b$

$a_0 = a+b$

$$\boxed{\text{Ker}(T) = \{(a+b) - 2ax - 3bx^2 : a,b \in R\}}$$

THEOREM 6.3: THE KERNEL IS A SUBSPACE OF *V*

The kernel of a linear transformation $T:V\rightarrow W$ is a subspace of the domain V.

Proof: We know that the ker(T) is a nonempty subset of V. [thm 6.1]

Let $\vec{u}$ and $\vec{v}$ be vectors in Ker(T), and let c be a scalar.

$$T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$$
$$= \vec{0}+\vec{0}$$
$$= \vec{0}. \checkmark$$

$$T(c\vec{u}) = cT(\vec{u})$$
$$= c\vec{0}$$
$$= \vec{0}. \checkmark$$

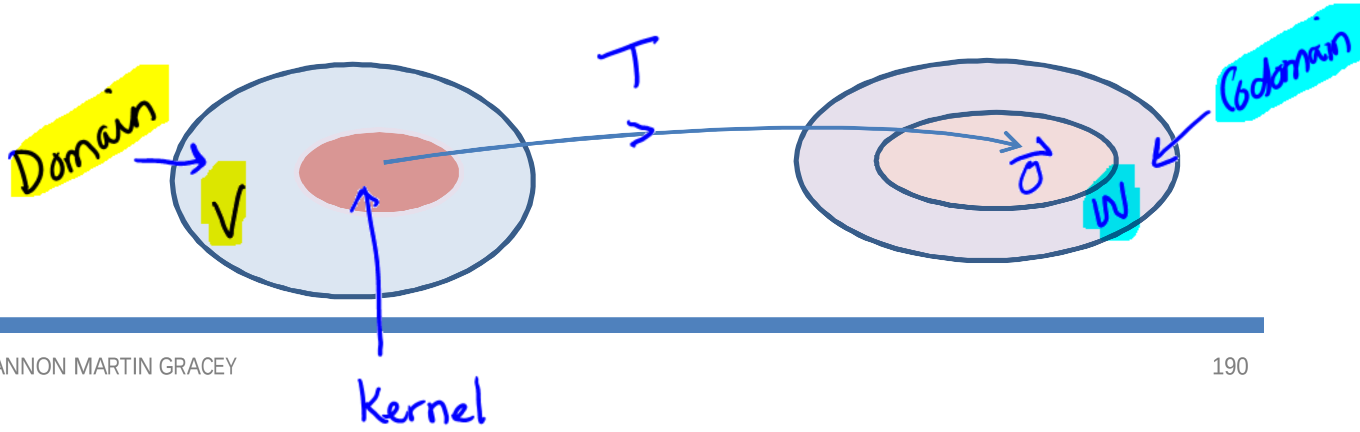
$\therefore$ Ker(T) is a subspace of V. //

THEOREM 6.3: COROLLARY

Let $T:R^n\rightarrow R^m$ be the linear transformation given by $T(\mathbf{x})=A\mathbf{x}$. Then the kernel of T is equal to the solution space of $A\vec{x}=\vec{0}$.

THEOREM 6.4: THE RANGE OF *T* IS A SUBSPACE OF *W*

The range of a linear transformation $T:V\rightarrow W$ is a subspace of W.



THEOREM 6.4: COROLLARY

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the Column space of ___A___ is equal to the ___range___ of ___T___.

Example 2: Let $T(\mathbf{v}) = A\mathbf{v}$ represent the linear transformation T. Find a basis for the kernel of T and the range of T.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$$

①

3 × 2

$R^2 \to R^3$

$$T(\vec{v}) = A\vec{v} = \vec{0}$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$v_1 + v_2 = 0$

$-v_1 + 2v_2 = 0$

$v_2 = 0$ so $v_1 = 0$

A Basis for Ker (T) is $\{(0,0)\}$.

② $$A^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{bmatrix}$$

A basis for the range (T) is $\{(1,-1,0), (1,2,1)\}$

$R^2 \to R^3$

DEFINITION OF RANK AND NULLITY OF A LINEAR TRANSFORMATION

Let $T: V \to W$ be a linear transformation. The dimension of the kernel of T is called the nullity of T and is denoted by $N(T)$. The dimension of the range of T is called the rank of T and is denoted by $\text{rank}(T)$.

THEOREM 6.5: SUM OF RANK AND NULLITY

Let $T: V \to W$ be a linear transformation from an n-dimensional vector space V into a vector space W. Then the sum of the dimension of the range and kernel is equal to the dimension of the domain. That is,

$$\text{rank}(T) + N(T) = n$$

or

$$\dim(\text{range}) + \dim(\text{Kernel}) = \dim(\text{domain})$$

Proof: T is represented by an $m \times n$ matrix. Assume $\text{rank}(A) = r$.

$$\begin{aligned}\text{rank}(T) &= \dim(\text{range}(T))\\ &= \dim(\text{column space})\\ &= \text{rank}(A)\\ &= r.\end{aligned}$$

We also know

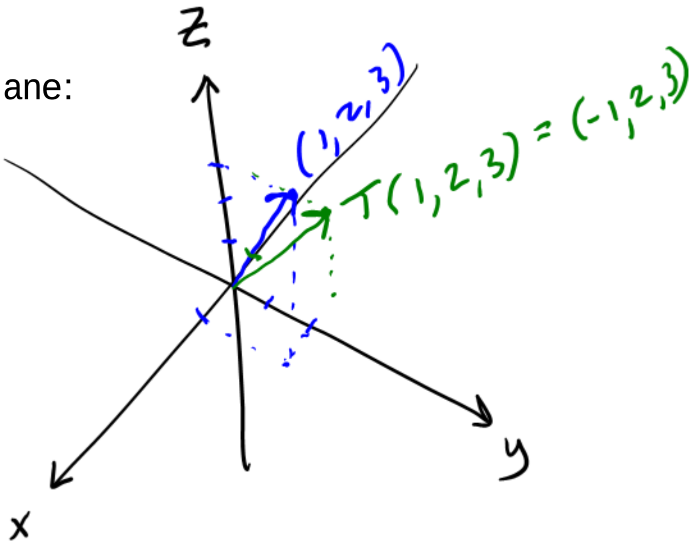
$$\begin{aligned}N(T) &= \dim(\text{Ker}(T)) \quad [\text{Thm } 4.17]\\ &= \dim(\text{solution space of } A\vec{x} = \vec{0})\\ &= n - r\end{aligned}$$

So $\text{rank}(T) + N(T) = (r) + (n-r) = n$. //

Example 4: Let $T: R^3 \to R^3$ be a linear transformation. Use the given information to find the nullity of T and give a geometric description of the kernel and range of T.

T is the reflection through the *yz*-coordinate plane:

$T(x, y, z) = (-x, y, z)$



$x = -x$
$y = y$
$z = z$

Ker (T) = (0,0,0) which is the ordered triple at the origin.
Range (T) = R^3

ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

If the zero vector is the only vector $\vec{v}$ such that $T(\vec{v}) = \vec{0}$, then T is one-to-one. A function $T: V \to W$ is called one-to-one when the preimage of every $\vec{w}$ in the range consists of a single vector. This is equivalent to saying that T is one-to-one if and only if, for all $\vec{u}$ and $\vec{v}$ in V, $T(\vec{u}) = T(\vec{v})$ implies that $\vec{u} = \vec{v}$.

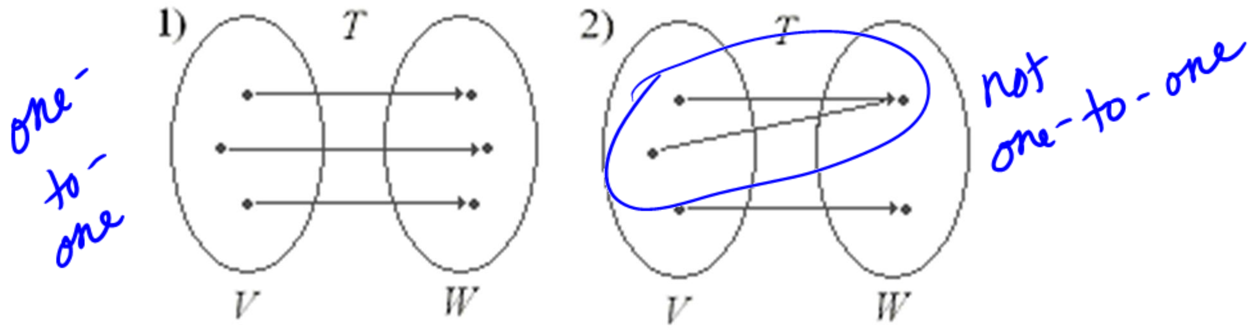


Figure 1

Example 3: Define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Find $\ker(T)$, $\text{nullity}(T)$, $\text{range}(T)$, and $\text{rank}(T)$.

3×5 $\quad m=3,\ n=5$

$$A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix} \quad R^3 \rightarrow R^5$$

Kernel

$T(\vec{v}) = A(\vec{v}) = \vec{0}$

$\vec{v}$ has to be 5×1

$$\begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccccc|c} 3 & -2 & 6 & -1 & 15 & 0 \\ 4 & 3 & 8 & 10 & -14 & 0 \\ 2 & -3 & 4 & -4 & 20 & 0 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$v_1 + 2v_3 + v_4 + v_5 = 0$

$v_2 + 2v_4 - 6v_5 = 0$

Let:

$v_3 = r$

$v_4 = s \rightarrow v_1 = -2r - s - t$

$v_5 = t \qquad v_2 = -2s + 6t$

$$\text{Ker}(T): \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -2r - s - t \\ -2s + 6t \\ r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Ker}(T) = \text{span}\left\{(-2,0,1,0,0), (-1,-2,0,1,0), (-1,6,0,0,1)\right\},\ N(T) = 3}$$

Range(T): The leading 1's in rref(A) are in the 1st and 2nd columns, so a basis for the range is $\{(3,4,2),(-2,3,-3)\}$.

$$\boxed{\begin{array}{l} \therefore\ \text{range}(T) = \text{span}\{(3,4,2),\ (-2,3,-3)\}. \\ \qquad \text{rank}(T) = 2 \end{array}}$$

THEOREM 6.6: ONE-TO-ONE LINEAR TRANSFORMATIONS

Let $T:V\to W$ be a linear transformation. Then T is one-to-one if and only if

___$\mathrm{Ker}(T)=\{\vec{0}\}$________________.

Proof: Suppose T is one-to-one.

$T(\vec{v})=\vec{0}$ has only one solution. $\therefore \mathrm{Ker}(T)=\vec{0}$,

Now suppose $\mathrm{Ker}(T)=\{\vec{0}\}$ and $T(\vec{u})=T(\vec{v})$.

$T(\vec{u}-\vec{v})=T(\vec{u})-T(\vec{v})=\vec{0}$. This implies that $\vec{u}-\vec{v}$ is in the Ker(T) and must equal $\vec{0}$. So $\vec{u}=\vec{v}$ which means that T is one-to-one. //

THEOREM 6.7: onto LINEAR TRANSFORMATIONS

Let $T:V\to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the _rank__________ of T is equal to the

____dimension_________ of W.

Proof:

THEOREM 6.8: ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation with vector spaces V and W, _both_______ of dimension *n.* Then T is one-to-one if and only if it is _onto_____.

Example 5: Determine whether the linear transformation is one-to-one, onto, or neither.

$T: R^2 \to R^2, T(x,y) = (x-y, y-x)$

$x = x-y = 0$
$y = y-x = 0$

$$\begin{array}{r} x - y = 0 \\ -x + y = 0 \\ \hline 0 = 0 \end{array}$$

$\boxed{\text{Neither}}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

range(T) = span$\{(1,-1)\}$

rank$(T) = 1$

Ker$(T) = \{(x,y) : x, y \in R\}$

so ∃ more than one solution, so T is not one-to-one.

~~Since T is not one-to-one, it's not onto~~

onto means rank(T) = dim(W).

rank$(T) \neq$ dim(W)

$1 \neq 2$

DEFINITION: ISOMORPHISM

A linear transformation $T:V \to W$ that is one-to-one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other.

THEOREM 6.9: ISOMORPHIC SPACES AND DIMENSION

Two finite dimensional vector spaces V and W are isomorphic if and only if they are of the same dimension.

Example 6: Determine a relationship among *m, n, j,* and *k* such that $M_{m,n}$ is isomorphic to $M_{j,k}$.

Section 6.3: MATRICES FOR LINEAR TRANSFORMATIONS

When you are done with your homework you should be able to…

- π Find the standard matrix for a linear transformation
- π Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation
- π Find the matrix for a linear transformation relative to a nonstandard basis

WHICH FORMAT IS BETTER? WHY?

Consider $T : R^3 \to R^3, T\left(x_1, x_2, x_3\right) = \left(4x_1 - x_2 - 5x_3, -2x_1 + x_2 + 6x_3, x_2 - 3x_3\right)$

and

$$T\left(\mathbf{x}\right) = A\mathbf{x} = \begin{bmatrix} 4 & -1 & -5 \\ -2 & 1 & 6 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What do you think?

1. easier to read
2. " " write
3. " " adapt to computer use

The key to representing a linear transformation $T : V \to W$ by a matrix is to determine how it acts on a basis for V. Once you know the image of every vector in the basis, you can use the properties of linear transformations to determine $T(\vec{v})$ for any $\vec{v}$ in V.

Do you remember the standard basis for R^n? Write this standard basis for R^n in column vector notation.

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \ldots, \underset{\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} \right\}$$

THEOREM 6.10: STANDARD MATRIX FOR A LINEAR TRANSFORMATION

Let $T : R^n \to R^m$ be a linear transformation such that, for the standard basis vectors $\mathbf{e}_i$ of R^n,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the standard matrix for T

Proof: $\vec{v} = [v_1 \ v_2 \ v_3 \ \cdots \ v_n]^T = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

$$\begin{aligned} T(\vec{v}) &= T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) \\ &= T(v_1\vec{e}_1) + T(v_2\vec{e}_2) + \cdots + T(v_n\vec{e}_n) \\ &= v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \cdots + v_n T(\vec{e}_n). \end{aligned}$$

[redacted]

$$A\vec{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(\vec{e}_1) + v_2 T(\vec{e}_2) + \cdots + v_n T(\vec{e}_n) \quad //$$

Example 1: Find the standard matrix for the linear transformation T.

$$T(x, y) = (4x + y, 0, 2x - 3y)$$

$\vec{e}_1 = (1, 0)$

$\vec{e}_2 = (0, 1)$

$T(1, 0) = (4, 0, 2) = T(\vec{e}_1)$

$T(0, 1) = (1, 0, -3) = T(\vec{e}_2)$

$$\boxed{A = \begin{bmatrix} 4 & 1 \\ 0 & 0 \\ 2 & -3 \end{bmatrix} \quad \text{standard matrix for } T}$$

Example 2: Use the standard matrix for the linear transformation T to find the image of the vector $\mathbf{v}$.

$T(x,y)=(x+y,x-y,2x,2y),\ \mathbf{v}=(3,-3)$

$\vec{e}_1=(1,0)$

$\vec{e}_2=(0,1)$

$T(\vec{e}_1)=T(1,0)=(1,1,2,0)$

$T(\vec{e}_2)=T(0,1)=(1,-1,0,2)$

$\rightarrow A=\begin{bmatrix}1&1\\1&-1\\2&0\\0&2\end{bmatrix},\ \vec{v}=(3,-3)$

$$A\vec{v}=\begin{bmatrix}1&1\\1&-1\\2&0\\0&2\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix}=\begin{bmatrix}0\\6\\6\\-6\end{bmatrix}$$

$$\boxed{T(3,-3)=(0,6,6,-6)}$$

Example 3: Consider the following linear transformation T:

T is the reflection through the *yz*-coordinate plane in $R^3 : T(x, y, z) = (-x, y, z),\ \mathbf{v} = (2,3,4)$.

a. Find the standard matrix A for the following linear transformation T.

$\vec{e}_1 = (1,0,0)$

$\vec{e}_2 = (0,1,0)$

$\vec{e}_3 = (0,0,1)$

$T(\vec{e}_1) = T(1,0,0) = (-1,0,0)$

$T(\vec{e}_2) = T(0,1,0) = (0,1,0)$

$T(\vec{e}_3) = T(0,0,1) = (0,0,1)$

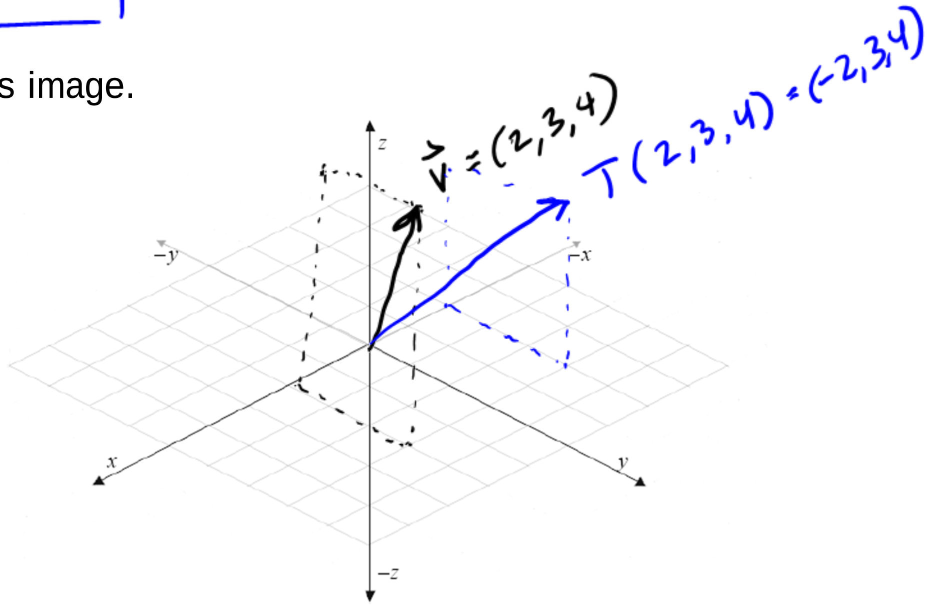
$$\boxed{A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

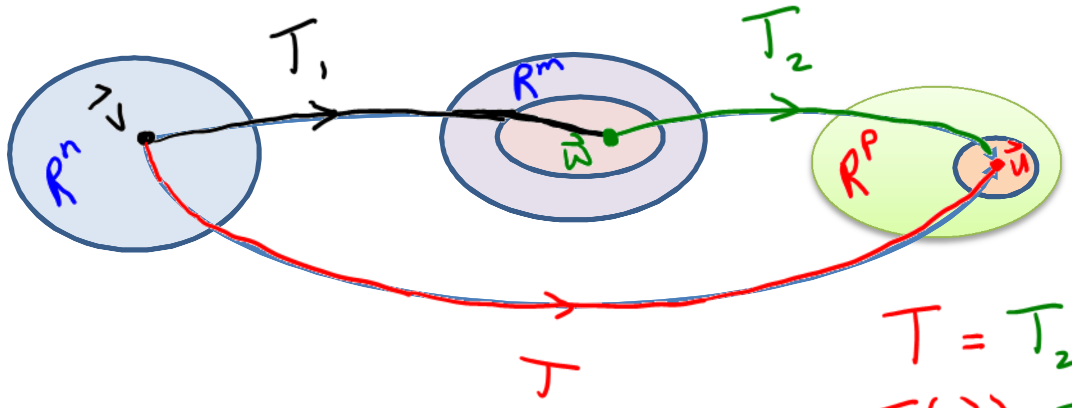
b. Use A to find the image of the vector $\mathbf{v}$.

$$A\vec{v} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

$$\boxed{T(2,3,4) = (-2,3,4)}$$

c. Sketch the graph of $\mathbf{v}$ and its image.





$T = T_2 \circ T_1$

$T(\vec{v}) = T_2\left[T_1(\vec{v})\right]$

THEOREM 6.11: COMPOSITION OF LINEAR TRANSFORMATIONS

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is given by the matrix product $A = A_2 A_1$.

Proof:

See video
or text

Example 4: Find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T = T_1 \circ T_2$.

$T_1 : R^2 \to R^3,\ T_1(x, y) = (x, y, y)$

$T_2 : R^3 \to R^2,\ T_2(x, y, z) = (y, z)$

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_1(\vec{e}_1) = T_1(1,0) = (1,0,0)$

$T_1(\vec{e}_2) = T_1(0,1) = (0,1,1)$

$T_2(\vec{e}_1) = T_2(1,0,0) = (0,0)$

$T_2(\vec{e}_2) = T_2(0,1,0) = (1,0)$

$T_2(\vec{e}_3) = T_2(0,0,1) = (0,1)$

① $T = T_2 \circ T_1 = A_2 A_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}$$

② $T = T_1 \circ T_2 = A_1 A_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

DEFINITION OF INVERSE LINEAR TRANSFORMATION

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are linear transformations such that for every $\mathbf{v}$ in R^n,

$$T_2\left[T_1(\vec{v})\right] = \vec{v} \text{ and } T_1\left[T_2(\vec{v})\right] = \vec{v}$$

then T_2 is called the __inverse________________ of T_1, and T_1 is said to be _invertible_________.

**Not every _linear________ transformation has an __inverse_________. If T_1___ is _invertible__________, however, the inverse is _unique_________ and is denoted by ___T_1^{-1}________.

THEOREM 6.12

Let is $T : R^n \to R^n$ be a linear transformation with a standard matrix A. Then the following conditions are equivalent.

1. T is invertible.
2. T is an isomorphism.
3. A is invertible.

If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Proof:

In video
or text

Example 5: Determine whether the linear transformation $T(x,y)=(x+y,x-y)$ is invertible. If it is, find its inverse.

$T(\vec{e}_1)=T(1,0)=(1,1)$

$T(\vec{e}_2)=T(0,1)=(1,-1)$

$$A=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad \det(A)=-1-1=-2\neq 0 \text{ so } A \text{ is invertible}$$

$$A^{-1}=\frac{1}{-2}\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{bmatrix}$$

$$T^{-1}(x,y)=\left(\tfrac{1}{2}x+\tfrac{1}{2}y,\tfrac{1}{2}x-\left(-\tfrac{1}{2}y\right)\right)=\boxed{\left(\tfrac{1}{2}x+\tfrac{1}{2}y,\tfrac{1}{2}x+\tfrac{1}{2}y\right)}$$

TRANSFORMATION MATRIX FOR NONSTANDARD BASES

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$.

If $T:V\to W$ is a linear transformation such that

$$\left[T(\mathbf{v}_1)\right]_{B'}=\begin{bmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{bmatrix},\ \left[T(\mathbf{v}_2)\right]_{B'}=\begin{bmatrix}a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{bmatrix},\ \ldots,\ \left[T(\mathbf{v}_n)\right]_{B'}=\begin{bmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{bmatrix},$$

then the $m\times n$ matrix whose n columns correspond to $\left[T(\mathbf{v}_1)\right]_{B'}$

$$A=\begin{bmatrix}a_{11} & \ldots & a_{1n}\\ \vdots & \vdots & \vdots\\ a_{m1} & \ldots & a_{mn}\end{bmatrix}$$

is such that $\left[T(\vec{v})\right]_{B'}$______________ for every $\vec{v}$ in V.

Example 6: Find $T(\mathbf{v})$ by using (a) the standard matrix, and (b) the matrix relative to B and B'.

$T: R^3 \to R^2,\ T(x,y,z) = (x-y, y-z),\ \mathbf{v} = (1,2,3),$

$B = \{(1,1,1),(1,1,0),(0,1,1)\},\ B' = \{(1,2),(1,1)\}$

a) $T(\vec{e}_1) = T(1,0,0) = (1,0)$

$T(\vec{e}_2) = T(0,1,0) = (-1,1)$

$T(\vec{e}_3) = T(0,0,1) = (0,-1)$

$$A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A'\vec{v} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\boxed{T(1,2,3) = (-1,-1)}$$

b) $T(1,1,1) = (0,0) = 0(1,2) + 0(1,1)$

$T(1,1,0) = (0,1) = 1(1,2) - 1(1,1)$

$T(0,1,1) = (-1,0) = 1(1,2) - 2(1,1)$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$(1,2,3) = c_1(1,1,1) + c_2(1,1,0) + c_3(0,1,1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$[\vec{v}]_B = (2,-1,1)$

$$[T(\vec{v})]_{B'} = A[\vec{v}]_B \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$T(1,2,3) = (0(1,2) - 1(1,1))$

$= (-1,-1)$

Example 6: Let $B = \left\{e^{2x}, xe^{2x}, x^2e^{2x}\right\}$ be a basis for a subspace of W of the space of continuous functions, and let D_x be the differential operator on W. Find the matrix for D_x relative to the basis B.

$$e^{2x} = e^{2x} + 0xe^{2x} + 0x^2e^{2x} \rightarrow D_x\left(e^{2x}\right) = 2e^{2x} + 0xe^{2x} + 0x^2e^{2x}$$

$$xe^{2x} = 0e^{2x} + xe^{2x} + 0x^2e^{2x} \rightarrow D_x\left(xe^{2x}\right) = 1e^{2x} + 2xe^{2x} + 0x^2e^{2x}$$

$$x^2e^{2x} = 0e^{2x} + 0xe^{2x} + x^2e^{2x} \rightarrow D_x\left(x^2e^{2x}\right) = 0e^{2x} + 2xe^{2x} + 2x^2e^{2x}$$

$$\boxed{[T]_B = A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}}$$

Section 6.4: TRANSITION MATRICES AND SIMILARITY

When you are done with your homework you should be able to…

- π Find and use a matrix for a linear transformation
- π Show that two matrices are similar and use the properties of similar matrices

A classical problem in linear algebra is determining whether it is possible to find a basis B such that the matrix for T relative to B is diagonal.

1. Matrix for T relative to B: A
2. Matrix for T relative to B': A'
3. Transition matrix from B' to B: P
4. Transition matrix from B to B': P^{-1}

$$\begin{tikzcd}
V & \left[\vec{v}\right]_B \arrow[r, "A", "(\text{basis } B)"'] & \left[T(\vec{v})\right]_B \arrow[d, "P^{-1}"] & V \\
V & \left[\vec{v}\right]_{B'} \arrow[u, "P"] \arrow[r, "A'", "(\text{basis } B')"'] & \left[T(\vec{v})\right]_{B'} & V
\end{tikzcd}$$

$$A'\left[\vec{v}\right]_{B'} = \left[T(\vec{v})\right]_{B'}$$

$$P^{-1}AP\left[\vec{v}\right]_{B'} = \left[T(\vec{v})\right]_{B'}$$

So, $A' = P^{-1}AP$

Example 1: Find the matrix A' relative to the basis B'.

$B = \{(1,0),(0,1)\}$

$T: R^2 \to R^2,\ T(x,y) = (x-2y, 4x),\ B' = \{(-2,1),(-1,1)\}$

$A' = P^{-1}AP$

Find P: $[B \vdots B']$

$T(\vec{e}_1) = T(1,0) = (1,4)$

$T(\vec{e}_2) = T(0,1) = (-2,0)$

$\{(1,4),(-2,0)\}$

$$[B \vdots B'] = \left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\;\middle\vdots\;\underbrace{\begin{matrix}-2 & -1\\ 1 & 1\end{matrix}}_{P}\right]$$

$$P = \begin{bmatrix}-2 & -1\\ 1 & 1\end{bmatrix} \quad P^{-1} = \frac{1}{-2-(-1)}\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 1 & 2\end{bmatrix} \quad A = \begin{bmatrix}1 & -2\\ 4 & 0\end{bmatrix}$$

$$A' = P^{-1}AP = \begin{bmatrix}-1 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & -2\\ 4 & 0\end{bmatrix}\begin{bmatrix}-2 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}12 & 7\\ -20 & -11\end{bmatrix}$$

Example 2: Let $B = \{(1,-1),(-2,1)\}$ and $B' = \{(-1,1),(1,2)\}$ be bases for R^2, $[\mathbf{v}]_{B'} = [1 \quad -4]^T$, and let $A = \begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to B.

a. Find the transition matrix P from B' to B.

$[B \vdots B'] \to [I_n \vdots P]$

$$\left[\begin{array}{cc|cc}1 & -2 & -1 & 1\\ -1 & 1 & 1 & 2\end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|cc}1 & -2 & -1 & 1\\ 0 & -1 & 0 & 3\end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc}1 & -2 & -1 & 1\\ 0 & 1 & 0 & -3\end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\;\middle\vdots\;\underbrace{\begin{matrix}-1 & -5\\ 0 & -3\end{matrix}}_{P}\right]$$

$$\boxed{P = \begin{bmatrix}-1 & -5\\ 0 & -3\end{bmatrix}} \quad P^{-1} = \frac{1}{3-0}\begin{bmatrix}-3 & 5\\ 0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 5/3\\ 0 & -1/3\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}}$$

$P = \begin{bmatrix}-1 & -5\\ 0 & -3\end{bmatrix}$ $P^{-1} = \begin{bmatrix}-1 & 5/3\\ 0 & -1/3\end{bmatrix}$ $A = \begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}$

b. Use the matrices P and A to find $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_{B'}$ where $[\mathbf{v}]_{B'} = [1 \quad -4]^T$.

2 ways to find the image of $\vec{v}$ under T relative to B':

$[\vec{v}]_B = P[\vec{v}]_{B'}$

$= \begin{bmatrix}-1 & -5\\ 0 & -3\end{bmatrix}\begin{bmatrix}1\\ -4\end{bmatrix}$

$= \boxed{\begin{bmatrix}19\\ 12\end{bmatrix}}$

1) $[T(\vec{v})]_{B'} = P^{-1}[T(\vec{v})]_B$

2) $[T(\vec{v})]_{B'} = \underset{\substack{\downarrow\\ P^{-1}AP}}{A'}\underbrace{[\vec{v}]_{B'}}_{\text{given in orig. problem}}$

$A' = \begin{bmatrix}-1 & 5/3\\ 0 & -1/3\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & -5\\ 0 & -3\end{bmatrix} = \begin{bmatrix}2 & 18\\ 0 & -1\end{bmatrix}$

$[T(\vec{v})]_{B'} = \begin{bmatrix}2 & 18\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -4\end{bmatrix} = \boxed{\begin{bmatrix}-70\\ 4\end{bmatrix}}$

DEFINITION OF SIMILAR MATRICES

For square matrices A and A' of order n, A' is said to be similar to A when there exists an invertible matrix P such that $A' = P^{-1}AP$.

THEOREM 6.13

Let A, B, and C be square matrices of order n. Then the following properties are true.

1. A is ___Similar___________________ to ___A___.
2. If A is similar to B, then ___B___ is ___Similar_____________ to ___A___.
3. If A is similar to B and B is similar to C, then ___A___ is ___similar_______________ to ___C___.

Proof:

1) $A = I_n A I_n$ (I_n is its own inverse) ✓

2) $PAP^{-1} = PP^{-1}BPP^{-1}$

$PAP^{-1} = I_n B I_n$

$PAP^{-1} = B$ → Let $Q = P^{-1}$, then $Q^{-1} = P$. So $B = Q^{-1}AQ$. ✓

3) Left to Student ☺

Example 3: Use the matrix P to show that A and A' are similar.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A' \overset{?}{=} P^{-1}AP$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

Yes!

DIAGONAL MATRICES

Diagonal matrices have many computational advantages over nondiagonal matrices.

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \qquad D^k = \begin{pmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{pmatrix}$$

Also, a diagonal matrix is its own transpose. Additionally, if all the diagonal elements are nonzero, then the inverse of a diagonal matrix is the matrix whose main diagonal elements are the reciprocals of corresponding elements in the original matrix. Because of these advantages, it is important to find ways (if possible) to choose a basis for V such that the transformation matrix is diagonal.

* need to complete

Example 4: Suppose A is the matrix for $T: R^3 \to R^3$ relative to the standard basis. Find the diagonal matrix A' for T relative to the basis B'.

Given: $B' = \{(1,1,-1), (1,-1,1), (-1,1,1)\}$ $\quad A = \begin{bmatrix} 3/2 & -1 & -1/2 \\ -1/2 & 2 & 1/2 \\ 1/2 & 1 & 5/2 \end{bmatrix}$

Need to know: $B = \{(1,0,0), (0,1,0), (0,0,1)\}$

$[B \vdots B'] \to [I_n \vdots P] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$ $\quad P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$[B' \vdots B] \to [I_n \vdots P^{-1}]$

$P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$A' = P^{-1}AP$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3/2 & -1 & -1/2 \\ -1/2 & 2 & 1/2 \\ 1/2 & 1 & 5/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}} = D$$

Example 5: Prove that if A is idempotent and B is similar to A, Then B is idempotent. (An nxn matrix is idempotent when $A = A^2$).

Proof: Evil Plan : Show that $B = B^2$

$B = P^{-1}AP$, P is an invertible matrix of order n.

$B^2 = (P^{-1}AP)^2$

$B^2 = (P^{-1}AP)(P^{-1}AP)$

$B^2 = P^{-1}A(PP^{-1})AP$

$B^2 = P^{-1}AI_nAP$

$B^2 = P^{-1}AAP$

$B^2 = P^{-1}A^2P$

$B^2 = P^{-1}AP$

$B^2 = B$ ✓

Section 7.1: EIGENVALUES AND EIGENVECTORS

When you are done with your homework you should be able to…

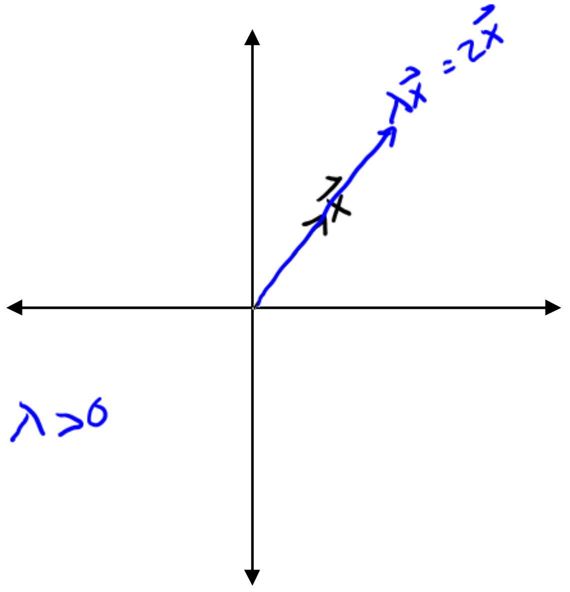
- Verify eigenvalues and corresponding eigenvectors
- Find eigenvectors and corresponding eigenspaces
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix
- Find the eigenvalues and eigenvectors of a linear transformation

THE EIGENVALUE PROBLEM

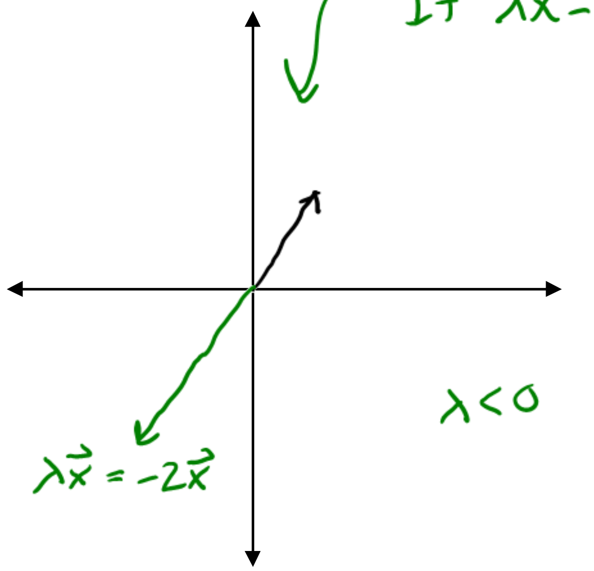
One of the most important problems in linear algebra is the **eigenvalue problem**.

When A is an $n \times n$, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a Scalar__________ multiple of $\mathbf{x}$? The scalar, denoted by _λ___ (lambda________), is called an eigenvalue____________ of the matrix A, and the nonzero vector $\mathbf{x}$ is called an eigenvector___________ of A corresponding to λ.

If $\lambda \vec{x} = 2\vec{x} \rightarrow \lambda = 2$



If $\lambda \vec{x} = -2\vec{x}$



DEFINITIONS OF EIGENVALUE AND EIGENVECTOR

Let A be an $n \times n$ matrix. The scalar ___λ___ is called an ___eigenvalue___ of A when there is a ___nonzero___ vector $\mathbf{x}$ such that ___$A\vec{x} = \lambda\vec{x}$___. The vector $\mathbf{x}$ is called an ___eigenvector___ of A corresponding to λ.

*Note that an eigenvector cannot be __$\vec{0}$__. Why not?

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{0} = \lambda\vec{0}$$
$$\vec{0} = \vec{0}$$

Example 1: Verify that λ_i is an eigenvector of A and that $\mathbf{x}_i$ is a corresponding eigenvector.

$$A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix},\ \lambda_1 = 2,\ \mathbf{x}_1 = (1,1),\ \lambda_2 = -3,\ \mathbf{x}_2 = (-4,1)$$

$$A\vec{x}_1 = \lambda_1\vec{x}_1$$

↙ I didn't realize we were given λ_1

$$\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore \lambda_1 = 2, \vec{x}_1 = (1,1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$

$$A\vec{x}_2 = \lambda_2\vec{x}_2$$
$$\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix} \overset{?}{=} -3\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 12 \\ -3 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 12 \\ -3 \end{bmatrix}$$

yes ✓

Example 2: Determine whether $\mathbf{x}$ is an eigenvector of A.

$$A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$$

a. $\mathbf{x} = (4,4)$

$$A\vec{x} = \lambda\vec{x}$$

$$\begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \lambda\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 28 \\ 28 \end{bmatrix} = 7\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

yes, $\lambda = 7$ and $\vec{x}$ is an eigenvector of A corresponding to $\lambda = 7$.

b. $\mathbf{x} = (-8,4)$

$$A\vec{x} = \lambda\vec{x}$$

$$\begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -8 \\ 4 \end{bmatrix} = \lambda\begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 64 \\ -32 \end{bmatrix} = \lambda\begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$-8\lambda = 64$ AND $4\lambda = -32$

$\lambda = -8$ ✓ $\qquad \lambda = -8$ ✓

$$\boxed{\text{yes!! } \lambda = -8}$$

c. $\mathbf{x} = (-4,8)$

$$\boxed{\text{NO}}$$

d. $\mathbf{x} = (5,-3)$

$$\boxed{\text{NO}}$$

THEOREM 6.11: EIGENVECTORS OF λ FORM A SUBSPACE

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

$$\{x : \vec{x} \text{ is an eigenvector of } A\} \cup \{\vec{0}\}$$

is a subspace of R^n. This subspace is called the eigenspace of λ.

Example 3: Find the eigenvalue(s) and corresponding eigenspace(s) of A.

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}, \quad \vec{v} = (x,y)$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x+ky \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x+ky = \lambda x \to x+ky = 1x \to ky = 0 \to x = x$$

$$ky = 0 \nearrow k=0 \text{ or } y=0$$

$$y = \lambda y \to y - \lambda y = 0 \to y(1-\lambda) = 0$$

$$y = 0 \text{ or } \lambda = 1$$

Conclusion:

$\lambda = 1$, the eigenspace is all vectors that lie on $y=0$, or the x-axis.

THEOREM 7.2: EIGENVALUES AND EIGENVECTORS OF A MATRIX

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\vec{x} = \vec{0}$.

* The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of A. When expanded to polynomial form, the polynomial

$$\det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

is called the characteristic polynomial of A. This definition tells you that the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A.

Example 4: Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \lambda I - A = \begin{bmatrix} \overset{+}{\lambda - 3} & -2 & -1 \\ \overset{-}{0} & \lambda & -2 \\ \overset{+}{0} & -2 & \lambda \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a)

$$\det(\lambda I - A) = 0$$

$$(\lambda - 3)[(\lambda^2 - 4)] - 0 + 0 = 0$$

$$(\lambda - 3)(\lambda^2 - 4) = 0$$

char. eq. in factored form

b) $\lambda = 3$ or $\lambda = \pm 2$

$$\boxed{\lambda_1 = 3, \lambda_2 = -2, \lambda_3 = 2}$$

i) $\lambda_1 = 3$:

$$3I - A = \begin{bmatrix} 3-3 & -2 & -1 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

$$(\lambda_1 I - A)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & -2 & -1 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = 0, x_3 = 0, x_1 = t, t \in \mathbb{R}$

The eigenvector of A corresponding to $\lambda_1 = 3$ is $\{(t,0,0) : t \in \mathbb{R}\}$

Next step... find the next 2 eigenvectors for λ_2, λ_3.

THEOREM 7.3: EIGENVALUES OF TRIANGULAR MATRICES

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main ___diagonal___.

Example 5: Find the eigenvalues of the triangular matrix.

$$\begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

$$\boxed{\lambda_1 = -5, \lambda_2 = 7, \lambda_3 = 3}$$

EIGENVALUES AND EIGENVECTORS OF LINEAR TRANSFORMATIONS

A number λ is called an eigenvalue of a linear transformation $T: V \to W$ when there is a nonzero vector $\vec{x}$ such that $T(\vec{x}) = \lambda\vec{x}$. The vector **x** is called an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

Example 6: Consider the linear transformation $T: R^n \to R^n$ whose matrix A relative to the standard basis is given. Find (a) the eigenvalues of A, (b) a basis for each of the corresponding eigenspaces, and (c) the matrix A' for T relative to the basis B', where B' is made up of the basis vectors found in part b).

$$A = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} \qquad \lambda I - A = \begin{bmatrix} \lambda+6 & -2 \\ -3 & \lambda+1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find λ_i:

$$\det(\lambda I - A) = 0$$
$$(\lambda+6)(\lambda+1) - 6 = 0$$
$$\lambda^2 + 7\lambda + \cancel{6} - \cancel{6} = 0$$
$$\lambda(\lambda+7) = 0$$
$$\lambda_1 = 0, \quad \lambda_2 = -7$$

b) $(\lambda I - A)\vec{x} = \vec{0}$

i: $\lambda_1 = 0$ $\quad \begin{bmatrix} 6 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 6 & -2 & 0 \\ -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 - \tfrac{1}{3}x_2 = 0$$
$$x_1 = \tfrac{1}{3}x_2$$
$$x_2 = 3t, \quad x_1 = t$$

eig. space corr. to $\lambda_1 = 0$ is

$\{(t, 3t) : t \in \mathbb{R}\}$

A basis for this space is:
let $t = 1 \to \{(1, 3)\}$

Next step... find a basis for $\lambda_2 = -7$

When you are done with your homework you should be able to…

- π Find the eigenvectors of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal
- π Find, for a linear transformation $T: V \to V$, a basis B for V such that the matrix for T relative to B is diagonal

DEFINITION OF A DIAGONALIZABLE MATRIX

An $n \times n$ matrix A is diagonalizable when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix _P__ such that _$P^{-1}AP$_______ is a diagonal matrix.

THEOREM 7.4: SIMILAR MATRICES HAVE THE SAME EIGENVALUES

If A and B are similar $n \times n$ matrices, then they have the same _eigenvalues___________.

Proof:

In text

Example 1: (a) verify that A is diagonalizable by computing $P^{-1}AP$, and (b) use the result of part (a) and Theorem 7.4 to find the eigenvalues of A.

$$A=\begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix},\ P=\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},\ P^{-1}=\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$P^{-1}AP \overset{?}{=} D \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = D$$

yes, A is diagonalizable and $\lambda_1 = 2$, $\lambda_2 = 4$.

THEOREM 7.5: CONDITION FOR DIAGONALIZATION

An $n \times n$ matrix A is diagonalizable if and only if it has n ___linearly___ ___independent___ eigenvectors.

Proof:

in text

Example 2: For each matrix A, find, if possible, a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & -2 & \lambda-2 \end{bmatrix}$$

$\lambda_1 = 4, \lambda_2 = \lambda_3 = 2$

$\lambda_1 = 4: (\lambda I - A)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_3 = 0$

$x_2 - x_3 = 0$

$x_1 = t, x_2 = t, x_3 = t$

$\vec{x}_1 = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{x}_1 = \{t(1,1,1) : t \in R\}$

$\lambda_2 = 2:$ $\begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = x_2 = 0 \\ x_3 = t \end{array}$$

$\vec{x}_2 = \{t(0,0,1) : t \in R\}$

So $(1,1,1)$ and $(0,0,1)$ are eigenvectors corresponding to $\lambda = 4$, $\lambda = 2$, respectively. Since $\exists$ only 2 linearly independent eigenvectors, by Thm 7.5, A is not diagonalizable.

STEPS FOR DIAGONALIZING AN $n \times n$ SQUARE MATRIX

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ for A (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

   If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

2. Let P be the $n \times n$ matrix whose columns consist of these eigenvectors.

   That is, $P = [\vec{p}_1 \ \vec{p}_2 \ \cdots \ \vec{p}_n]$.

3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal (and zeros elsewhere). Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

$$D = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix}$$

THEOREM 7.6: SUFFICIENT CONDITION FOR DIAGONALIZATION

If an $n \times n$ matrix A has __n__ __distinct__ eigenvalues, then the corresponding eigenvectors are __linearly__ __independent__ and A is __diagonalizable__.

Proof:

in text

Example 3: Find the eigenvalues of the matrix and determine whether there is a sufficient number to guarantee that the matrix is diagonalizable.

$A=\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$

Since A is triangular, the eigenvalues are $\lambda_1=\lambda_2=2$. Since there is only 1 distinct eigenvalue, A is not diagonalizable.

Example 4: Find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

$T: R^3 \to R^3 : T(x, y, z) = (-2x+2y-3z, 2x+y-6z, -x-2y)$

Section 7.3: SYMMETRIC MATRICES AND ORTHOGONAL DIAGONALIZATION

When you are done with your homework you should be able to…

- Recognize, and apply properties of, symmetric matrices
- Recognize, and apply properties of, orthogonal matrices
- Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A

SYMMETRIC MATRICES

Symmetric matrices arise more often in applications than any other major class of matrices. The theory depends on both symmetry and orthogonality. For most matrices, you need to go through most of the diagonalization process to ascertain whether a matrix is diagonalizable. We learned about one exception, a triangular matrix, which has eigenvalue entries on the main diagonal. Another type of matrix which is guaranteed to be diagonalizable is a symmetric matrix.

DEFINITION OF SYMMETRIC MATRIX

A square matrix A is symmetric when it is equal to its transpose: $A = A^T$.

Example 1: Determine which of the matrices below are symmetric.

$$\textcircled{A} = \begin{bmatrix} -2 & 5 \\ 5 & 1 \end{bmatrix}, \textcircled{B} = \begin{bmatrix} 6 & 5 & 4 \\ 5 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \textcircled{D} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 7 & 1 & 0 \\ 3 & 1 & 7 & 2 \\ 4 & 0 & 2 & 5 \end{bmatrix}$$

Example 2: Using the diagonalization process, determine if A is diagonalizable. If so, diagonalize the matrix A.

$$A=\begin{bmatrix}6 & -1\\ -1 & 5\end{bmatrix}$$

$$\lambda I-A=\begin{bmatrix}\lambda-6 & 1\\ 1 & \lambda-5\end{bmatrix}$$

$$|\lambda I-A| = (\lambda-6)(\lambda-5)-1$$

$$0 = \lambda^2-11\lambda+29$$

$$\lambda=\frac{11\pm\sqrt{121-116}}{2}$$

$$\lambda=\frac{11\pm\sqrt{5}}{2}$$

$$\lambda_1=\frac{11-\sqrt{5}}{2}\approx 4.4$$

$$\lambda_2=\frac{11+\sqrt{5}}{2}\approx 6.6$$

$$P=\begin{bmatrix}0.618 & -1.618\\ 1 & 1\end{bmatrix}\quad P^{-1}\doteq\begin{bmatrix}0.447 & 0.724\\ -0.447 & 0.276\end{bmatrix}$$

$$P^{-1}AP\doteq\begin{bmatrix}4.382 & 0\\ 0 & 6.618\end{bmatrix}=D$$

From Octave:

```
D =

   4.3820e+00  -3.3989e-05
   3.3989e-05   6.6180e+00
```

$$\lambda_1=\frac{11-\sqrt{5}}{2}\qquad \lambda_1 I-A=\begin{bmatrix}\frac{11-\sqrt{5}}{2}-6 & 1\\ 1 & \frac{11-\sqrt{5}}{2}-5\end{bmatrix}$$

```
A = [(11-sqrt(5))/2-6 1; 1 (11-sqrt(5))/2-5]
A =

  -1.61803   1.00000
   1.00000  -0.61803

rref(A)
ans =

   1.00000  -0.61803
   0.00000   0.00000
```

$$x_1-0.618x_2\doteq 0$$

$$x_1\doteq 0.618t,\ x_2=t$$

$$\vec{x}_1=\{(0.618t,t):t\in R\}$$

$$\vec{p}_1=(0.618,1)^T$$

$$\lambda_2=\frac{11+\sqrt{5}}{2}:\ \lambda_2 I-A=\begin{bmatrix}\frac{11+\sqrt{5}}{2}-6 & 1\\ 1 & \frac{11+\sqrt{5}}{2}-5\end{bmatrix}$$

```
A = [(11+sqrt(5))/2-6 1; 1 (11+sqrt(5))/2-5]
A =

   0.61803   1.00000
   1.00000   1.61803

rref(A)
ans =

   1.00000   1.61803
   0.00000   0.00000
```

$$x_1+1.618x_2\doteq 0$$

$$x_1\doteq -1.618t,\ x_2=t$$

$$\vec{x}_2\doteq\{(-1.618t,t):t\in R\}$$

$$\vec{p}_2=(-1.618,1)^T$$

Note: When we found rref of each $\lambda_i I-A$, in class we only input the 4.4 and the 6.6 instead subtracting the existing scalars 6 and 5.

THEOREM 7.7: PROPERTIES OF SYMMETRIC MATRICES

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is diagonalizable.
2. All eigenvalues of A are real.
3. If λ is an eigenvalue of A with multiplicity K, then λ has K linearly independent eigenvectors. That is, the eigenspace of λ has dimension K.

Proof of Property 1 (for a 2 x 2 symmetric matrix):

in text

Example 3: Prove that the symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Example 4: Find the eigenvalues of the symmetric matrix. For each eigenvalue, find the dimension of the corresponding eigenspace.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \overset{+}{\lambda-2} & 1 & 1 \\ \overset{-}{1} & \lambda-2 & 1 \\ \overset{+}{1} & 1 & \lambda-2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0$$

$$(\lambda-2)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & \lambda-2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ \lambda-2 & 1 \end{vmatrix} = 0$$

$$(\lambda-2)\left[(\lambda-2)^2 - 1\right] - \left[(\lambda-2)-1\right] + \left[1-(\lambda-2)\right] = 0$$

$$(\lambda-2)^3 - (\lambda-2) - (\lambda-2) + 1 + 1 - (\lambda-2) = 0$$

$$(\lambda-2)^3 - 3(\lambda-2) + 2 = 0$$

$$(\lambda-2)^3 - 3\lambda + 6 + 2 = 0$$

$$1(\lambda)^3(-2)^0 + 3(\lambda)^2(-2)^1 + 3(\lambda)^1(-2)^2 + 1(\lambda)^0(-2)^3 - 3\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 - 3\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$\lambda_1 = 0$ or $(\lambda-3)^2 = 0$

$\lambda_2 = 3$

$\lambda_1 = 0,\ \lambda_2 = 3$

The dimension of the eigenspace corresponding to $\lambda_1 = 0$ is 1

" " " " " " " $\lambda_2 = 3$ is 2

DEFINITION OF AN ORTHOGONAL MATRIX

A square matrix P is orthogonal ______________ when it is invertible ______________ and when $P^{-1} = P^T$ ______________.

THEOREM 7.8: PROPERTY OF ORTHOGONAL MATRICES

An $n \times n$ matrix P is orthogonal if and only if its column ____________________ vectors form an orthonormal ____________________ set.

Proof:

In text

Example 5: Determine whether the matrix is orthogonal. If the matrix is orthogonal, then show that the column vectors of the matrix form an orthonormal set.

$$A = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$$AA^{-1} = I_3$$

$$A^2 = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

So A is invertible and $A = A^T$.

$$A = A^T$$

Since $A^{-1} = A^T$ and A is invertible, A is orthogonal.

$$\vec{p}_1 = \begin{bmatrix} -4/5 \\ 0 \\ 3/5 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$\vec{p}_1 \cdot \vec{p}_2 = 0$

$\vec{p}_1 \cdot \vec{p}_3 = 0$

$\vec{p}_2 \cdot \vec{p} = 0$

mutually orthogonal

$\|\vec{p}_1\| = \sqrt{16/25 + 0 + 9/25} = 1$

$\|\vec{p}_2\| = 1$ So, $\vec{p}_i$ are unit vectors

$\|\vec{p}_3\| = 1$

$$\boxed{\therefore \text{The column vectors of } A \text{ are orthonormal.}}$$

THEOREM 7.9: PROPERTY OF SYMMETRIC MATRICES

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

Proof: Suppose Is an nxn symmetric matrix with distinct eigenvalues, λ_1 and λ_2. Let $\vec{x}_1$ and $\vec{x}_2$ be eigenvectors corresponding to λ_1 and λ_2, respectively.

$$\begin{aligned}
\lambda_1(\vec{x}_1 \cdot \vec{x}_2) &= (\lambda_1 \vec{x}_1) \cdot \vec{x}_2 \\
&= (A\vec{x}_1) \cdot \vec{x}_2 \\
&= (A\vec{x}_1)^T \cdot \vec{x}_2 \\
&= (\vec{x}_1^T A^T) \cdot \vec{x}_2 \\
&= (\vec{x}_1^T A) \cdot \vec{x}_2 \\
&= \vec{x}_1^T \cdot (A\vec{x}_2) \\
&= \vec{x}_1 \cdot (\lambda_2 \vec{x}_2) \\
&= \lambda_2 (\vec{x}_1 \cdot \vec{x}_2).
\end{aligned}$$

Since $\lambda_1(\vec{x}_1 \cdot \vec{x}_2) = \lambda_2(\vec{x}_1 \cdot \vec{x}_2)$

$$\lambda_1(\vec{x}_1 \cdot \vec{x}_2) - \lambda_2(\vec{x}_1 \cdot \vec{x}_2) = 0$$

$$(\vec{x}_1 \cdot \vec{x}_2)\underbrace{(\lambda_1 - \lambda_2)}_{\lambda_1 \neq \lambda_2} = 0$$

so, $(\vec{x}_1 \cdot \vec{x}_2) = 0$.

$\therefore$ $\vec{x}_1$ and $\vec{x}_2$ are orthogal. //

THEOREM 7.10: FUNDAMENTAL THEOREM OF SYMMETRIC MATRICES

Let A be an $n \times n$ matrix. Then A is __orthogonally__ __diagonalizable__ and has __real__ eigenvalues if and only if A is __symmetric__.

Proof:

In text

STEPS FOR DIAGONALIZING A SYMMETRIC MATRIX

Let A be an $n \times n$ symmetric matrix.

1. Find all eigenvalues of A and determine the multiplicity of each.
2. For each eigenvalue of multiplicity $k \geq 2$, find a unit eigenvector. That is, find any eigenvector and then normalize it.
3. For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply the Gram-Schmidt orthonormalization process.
4. The results of steps 2 and 3 produce an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = D$ will be diagonal. The main entries of D are the eigenvalues of A.

Example 5: Find a matrix P such that P^TAP orthogonally diagonalizes A. Verify that P^TAP gives the proper diagonal form.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & \underset{1}{0} & 0 \end{bmatrix} \qquad \lambda I - A = \left[\begin{array}{ccc} + & - & + \\ \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{array}\right]$$

$$\det(\lambda I - A) = 0$$

$$\lambda(\lambda^2 - 1) + (-\lambda - 1) - (1 + \lambda) = 0$$

$$\lambda^3 - \lambda - \lambda - 1 - 1 - \lambda = 0$$

$$\lambda^3 - 3\lambda - 2 = 0$$

$\lambda_1 = -1$ (mult 2)

$\lambda_2 = 2$

$$\lambda_1 = -1 : \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $(\lambda I - A)\vec{x}_1 = \vec{0}$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 = -x_2 - x_3 = -s - t$$

$$\vec{x}_1 = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

↓ ↘

$(-1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 0, 1/\sqrt{2})$

$$\lambda_2 = 2 : \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 \quad - x_3 = 0 \rightarrow x_1 = x_3 = t$

$x_2 - x_3 = 0 \rightarrow x_2 = x_3 = t$

$$\vec{x}_2 = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = \vec{u}_3$$

$\vec{u}_1 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$

$\vec{u}_2 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$

Let $\vec{w}_1 = \vec{u}_1$ and apply G.S.

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$P^TAP \overset{?}{=} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

yes.

Example 6: Prove that if a symmetric matrix A has only one eigenvalue λ, then $A = \lambda I$.

Pf: Let A be a symmetric matrix with only 1 eigenvalue, λ.
Since A is symmetric, the exists a P such that $P^{-1}AP = D$
where D is a diagonal matrix.

$$PP^{-1}AP = PD$$
$$APP^{-1} = PDP^{-1}$$
$$A = PDP^{-1}$$
$$A = P(\lambda I)P^{-1}$$
$$A = P\lambda P^{-1} I$$
$$A = PP^{-1}\lambda I$$
$$A = \lambda I \quad //$$

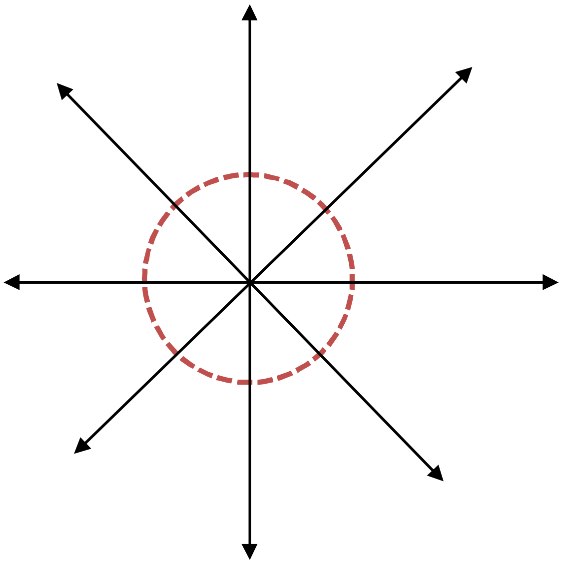
When you are done with your homework you should be able to…

- π Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric

QUADRATIC FORMS

Every conic section in the *xy*-plane can be written as :

If the equation of the conic has no *xy*-term (______________), then the axes of the graphs are parallel to the coordinate axes. For second-degree equations that have an *xy*-term, it is helpful to first perform a ____________________ of axes that eliminates the *xy*-term. The required rotation angle is $\cot 2\theta = \frac{a-c}{b}$. With this rotation, the standard basis for R^2, ______________________ is rotated to form the new basis _______________________________________.

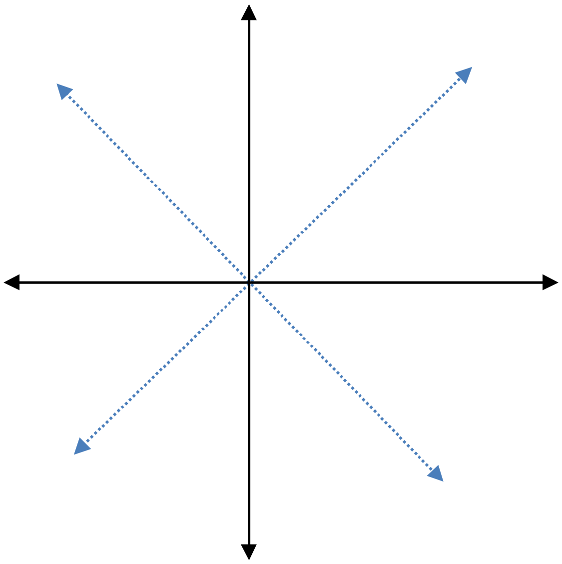


Example 1: Find the coordinates of a point (x, y) in R^2 relative to the basis $B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$.

ROTATION OF AXES

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form $a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$ by rotating the coordinate axes counterclockwise through the angle θ, where θ is defined by $\cot 2\theta = \frac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Example 2: Perform a rotation of axes to eliminate the *xy*-terms in $5x^2-6xy+5y^2+14\sqrt{2}x-2\sqrt{2}y+18=0$. Sketch the graph of the resulting equation.



_________________________ and __________________________ can be used to solve the rotation of axes problem. It turns out that the coefficients *a'* and *c'* are eigenvalues of the matrix

The expression ________________________ is called the __________________ form associated with the quadratic equation

and the matrix ____ is called the ________________ of the ________________ form. Note that ____ is ______________________. Moreover, ___ will be _________________________ if and only if its corresponding quadratic form has no ______ term.

Example 3: Find the matrix of quadratic form associated with each quadratic equation.

a. $x^2 + 4y^2 + 4 = 0$

b. $5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$

Now, let's check out how to use the matrix of quadratic form to perform a rotation of axes.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as follows:

If _______________, then no ______________________ is necessary. But if ______________, then because ___ is symmetric, you may conclude that there exists an _____________________ matrix ____ such that _________________ is diagonal. So, if you let

then it follows that ___________, and

The choice of ___ must be made with care. Since ___ is orthogonal, its determinant will be ___________. If P is chosen so that $|P| = 1$, then P will be of the form

where θ gives the angle of rotation of the conic measured from the ___________ *x*-axis to the positive *x'*-axis.

PRINCIPAL AXES THEOREM

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by ____________________ eliminates the *xy*-term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A. That is

where λ_1 and λ_2 are eigenvalues of A. The equation of the rotated conic is given by

Example 4: Use the Principal Axes Theorem to perform a rotation of axes to eliminate the *xy*-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system.

$$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$$